Newtonian Physics

A Self-Contained Introduction with Mathematical Foundations

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Preface

Physics tells stories about change. Newtonian physics, in particular, gives a precise language to describe how things move and why. This book aims to be rigorous where it matters and friendly everywhere: figures first, formulas second, with short proofs or precise definitions when they sharpen understanding.

How to Read

Skim a section for its figures and callouts to catch the idea. Then read the surrounding narrative for intuition and the equations for precision. Each chapter starts with Learning Objectives and a Symbols-at-a-Glance box, and ends with a Summary, a short *Where We're Heading Next*, a Common Pitfalls callout, and a Try in 60 seconds mini-checklist. Use the guide below to choose how deep to go.

Guide for Readers

Pick a path that matches your goal and time.

Conceptual Tour (intuition first). Follow Parts I–VI for the story of motion, relying on figures, analogies, and summaries. Skim the heavier ODE and numerical derivations (e.g., Chapter 2 Sections 2.4–2.5; Chapter 7 Section 7.5). Skip Part VII and treat the appendices as optional reference, glancing at the glossary when a symbol or term is new.

Mathematical Deep Dive (rigor and computation). Read everything in Parts I–VII, including the ODE structure and numerical preview in Chapter 2, the force-to-motion component work in Chapter 7, and the modeling and simulation focus in Chapters 15 and 16. Pair chapters with the math appendices: start with Appendices A–B (calculus, vectors), use Appendix C for gradients/curl and the conservative field test $\nabla \times \mathbf{F} = \mathbf{0}$ and $\mathbf{F} = -\nabla U$, and lean on Appendices D–E for ODE stability, error scaling, and why symplectic Euler behaves better on energy.

Style and Approach

We keep a light-hearted tone while treating the math with respect. Figures and short callouts lead the way; precise equations and compact proofs follow where they add clarity. We use the International System of Units (SI), check dimensions as a habit, and favor analogies and worked examples that tie the symbols back to everyday experience. Exercises mix quick conceptual checks with approachable real-world tasks.

What This Book Covers

The book is self-contained and spans the essentials of Newtonian mechanics—concepts, methods, and a compact math appendix. Here is a preview so you can choose your path:

- **Part I** builds intuition and language: Chapter 1 frames the scope and scales; Chapter 2 develops graphs, vectors, calculus ideas, ODEs, and a numerical preview.
- **Part II** moves along a line: Chapter 3 ties slopes/areas to $x(t), v(t), a(t)$; Chapter 4 links forces to motion and introduces simple numerical updates.
- **Part III** adds dimensions: Chapter 5 develops vector kinematics; Chapter 6 applies the toolkit to projectile and circular motion.
- **Part IV** unifies forces and energy: Chapter 7 (FBDs, components, friction/drag/tension) and Chapter 8 (work, kinetic energy, power) plus Chapter 9 (potential energy and conservation).
- **Part V** treats many-particle systems and rotation: Chapter 10 (COM, momentum, collisions) and Chapter 11 (torque, rotational dynamics, energy, angular momentum).
- **Part VI** samples gravity, oscillations, and a taste of continua: Chapters 12 to 14.
- **Part VII** adds cross-cutting methods: Chapter 15 (dimensions, scaling) and Chapter 16 (numerical time-stepping and stability/accuracy intuition).
- **Appendices** collect just the math we actually use: calculus (definitions + visuals), vectors/linear algebra, multivariable essentials (gradient/divergence/curl; line integrals), ODEs (separable/linear; oscillations; slope fields), and numerics (Euler flavors; error/stability/energy checks).

Who This Book Is For

Learners with high-school algebra/trigonometry who want a clear, visual path into mechanics; engineers and technically minded professionals (including quantitative analysts, financial modellers, algorithmic traders, and researchers) who want a compact, simulation-ready refresher with reproducible figures; and autodidacts who prefer a friendly tone without sacrificing correctness.

Objectives

By the end, you should be able to model motion with clean diagrams and equations; choose between force- and energy-based methods; reason about orders of magnitude; and make pragmatic numerical predictions while checking accuracy and stability.

Notation and Conventions

We use the International System of Units (SI) throughout. Scalars appear in italic (e.g., m, t); vectors in bold lower-case (e.g., $\boldsymbol{v}, \boldsymbol{a}$); matrices and tensors in bold upper-case (e.g., $\boldsymbol{R}$). Unit vectors carry hats (e.g., $\hat{\boldsymbol{e}}_x$). Time derivatives use an overdot (e.g., $\dot{x}$), and generic derivatives use prime when context is clear ($f'(x)$).

Quick Reference

- Position: x (1D) or $\boldsymbol{r} = (x, y, z)$; velocity $v = \dot{x}$, $\boldsymbol{v} = \dot{\boldsymbol{r}}$; acceleration $a = \dot{v}$, $\boldsymbol{a} = \dot{\boldsymbol{v}}$.
- Force F or $\boldsymbol{F}$; mass m; weight $W = mg$; gravitational acceleration $g \approx 9.81\,\mathrm{m/s^2}$ near Earth.
- Energy: kinetic $K = \frac{1}{2}mv^2$; potential U (context-dependent); power $P = \boldsymbol{F} \cdot \boldsymbol{v}$ (see Chapter 8).
- Products: dot $\boldsymbol{a} \cdot \boldsymbol{b}$ (alignment/projection); cross $\boldsymbol{a} \times \boldsymbol{b}$ (area/torque; right-hand rule).
- Matrices: $\boldsymbol{a}' = \boldsymbol{R}\,\boldsymbol{a}$ for rotations in 2D/3D (see Appendix B and Chapter 11).

Angles, Rotation, and Systems

- Angles θ are in *radians* unless stated; angular velocity $\omega = \dot{\theta}$; angular acceleration $\alpha = \dot{\omega}$.
- Torque τ about a specified axis; moment of inertia I about that axis; rotational energy $K_{\text{rot}} = \frac{1}{2}I\omega^2$; angular momentum $L = I\,\omega$ (fixed axis).
- Momentum $\boldsymbol{p} = m\,\boldsymbol{v}$; center of mass $\boldsymbol{R} = \sum m_i \boldsymbol{r}_i / \sum m_i$ with $M\,\ddot{\boldsymbol{R}} = \sum \boldsymbol{F}_{\text{ext}}$.

Dimensionless Numbers and Parameters

- Reynolds number $\mathrm{Re} = \dfrac{\rho v L}{\eta}$; Froude number $\mathrm{Fr} = \dfrac{v}{\sqrt{gL}}$ (see Chapter 15).
- Coefficient of restitution e (1D collisions); damping ratio ζ and quality factor $Q = 1/(2\zeta)$ (Chapter 13).
- Gravitational parameter $\mu = GM$; specific energy $\varepsilon = \frac{1}{2}v^2 + \Phi$ with $\Phi = -\mu/r$ (Chapter 12).
- Numerical step Δt; $t_n = n\,\Delta t$ (Chapter 16 and appendix E).

Operators and Fields

- Gradient ∇f points uphill; directional derivative $D_{\hat{\boldsymbol{u}}} f = \nabla f \cdot \hat{\boldsymbol{u}}$.
- Divergence $\nabla \cdot \boldsymbol{F}$ measures sources/sinks; curl $\nabla \times \boldsymbol{F}$ measures swirl (see Appendix C).
- Line integral along a path $\mathcal{C}$: $\int_{\mathcal{C}} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r}$; conservative fields are gradients of potentials.

Dimensions and Units

Dimensional analysis is used routinely as a check:

$$[x] = \mathrm{L}, \quad [t] = \mathrm{T}, \quad [v] = \mathrm{L\,T^{-1}}, \quad [a] = \mathrm{L\,T^{-2}}, \quad [F] = \mathrm{M\,L\,T^{-2}}.$$

We spell out acronyms on first use (e.g., International System of Units (SI)). Tables and figures report units in brackets, e.g., x [m].

Acknowledgments

With gratitude, I acknowledge the artificial intelligence research community and OpenAI for advancing tools such as Codex and GPT-5. Their work has made projects like this book both feasible and joyful. It is a privilege to witness—and to contribute during—this period of rapid progress.

I am not a physicist by formal training, but I have been fascinated by physics since school. Leveraging modern AI systems lets me write—and learn from—books I wish already existed. The style and presentation reflect my preferences and experience: figures first, clear prose, and just enough formalism to sharpen intuition.

I would be grateful for feedback on any of my AI-powered projects. You can find ways to get in touch at `https://linktr.ee/dyjh`.

Disclaimer. This book is intended solely for educational and illustrative purposes. It has not been formally peer-reviewed or vetted as a textbook, and it may contain inaccuracies or omissions. It is shared in the hope that readers may benefit from a different way of presenting these topics to a broader audience, but it should not be regarded as an authoritative reference. Readers are encouraged to cross-check important results and interpretations with standard physics texts or qualified instructors.

Contents

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Part I

Conceptual & Mathematical Foundations

Part I Overview

This opening part builds intuition and language. Chapter 1 frames the scope of Newtonian mechanics with everyday examples, units, and the structure of theories. Chapter 2 develops the mathematical toolkit—functions and graphs, vectors, derivatives/integrals, and ordinary differential equations (ODEs)—with visual and practical analogies.

Chapter 1
What Is Newtonian Physics?

Welcome to a rigorous yet light-hearted tour of classical mechanics. We will keep the math honest, the plots neat, and the jokes strictly inertial.

Learning Objectives

After this chapter you should be able to describe what "classical" means in context, recognize when Newtonian ideas apply, read our basic notation and units, and explain in one paragraph how a physical theory connects states, laws, and solutions.

Symbols at a Glance

t time; x position; $v = \dot{x}$ velocity; $a = \dot{v}$ acceleration; m mass; F force; $g \approx 9.81\,\mathrm{m/s^2}$ near Earth; energy K, U; momentum p.

Analogy: Stories of Change

Physics tells stories about change. Each story has a cast (objects with mass), a script (laws), and a timeline (initial conditions evolving in time). Reading and writing those stories is the goal of this book.

At a Glance

Newtonian physics studies how physical systems evolve in time when governed by simple laws. We emphasize clear mathematical structure, friendly figures, and practical intuition.

1.1 The Scope of Newtonian Mechanics

Classical mechanics applies when speeds are small compared to the speed of light and when quantum effects can be neglected. It is the theory of everyday motion: bicycles and buses, tennis balls and elevators. Prototypical models include point particles (e.g., a marble), rigid bodies (e.g., a book or a door), and idealized continua (e.g., a uniform rod or a beam).

In practice, we choose a model that captures the essence of a question and ignores details that do not matter at the scale of interest. A thrown ball, for example, can be treated as a point mass for its flight, but as a rotating rigid body when spin matters.

When Newtonian Ideas Break

Relativity matters when speeds approach the speed of light; quantum effects matter when actions are comparable to Planck's constant. Everyday motion on human scales sits comfortably in the Newtonian regime. When in doubt, estimate: if a commuter train travels at $v \approx 50\,\mathrm{m/s}$, then $v/c \approx 1.7 \times 10^{-7}$—firmly classical.

A Friendly Analogy

Think of mechanics like a recipe: the *ingredients* are states (positions and velocities), the *instructions* are the laws (forces), and the *cake* is the motion you observe after time evolves.

1.2 Basic Physical Quantities

We use position and time to describe motion, mass to measure inertia, and derived quantities like velocity, acceleration, force, energy, momentum, and angular momentum to analyze behavior. These quantities form a small vocabulary that we will reuse in many contexts.

Analogy: Casting a Movie

Think of a motion problem like casting a short movie. The *actors* are the objects (with mass), the *script* is the force law, and the *cinematography* is the coordinate system you choose to tell the story. The same scene can be filmed from different angles (coordinates); the plot stays the same, but some shots make the action clearer.

Instant Snapshot vs. Flipbook

An "instant" (state) is like a single photograph; a trajectory is like a flipbook. Newtonian laws tell you how to go from one picture to the next. A good model lets you predict the next frame so well that watching the flipbook feels inevitable.

A First Visualization: Constant Acceleration

As previewed in Figures 1.1 and 1.2, we sketch position and velocity under constant acceleration (think of a car smoothly pressing the accelerator).

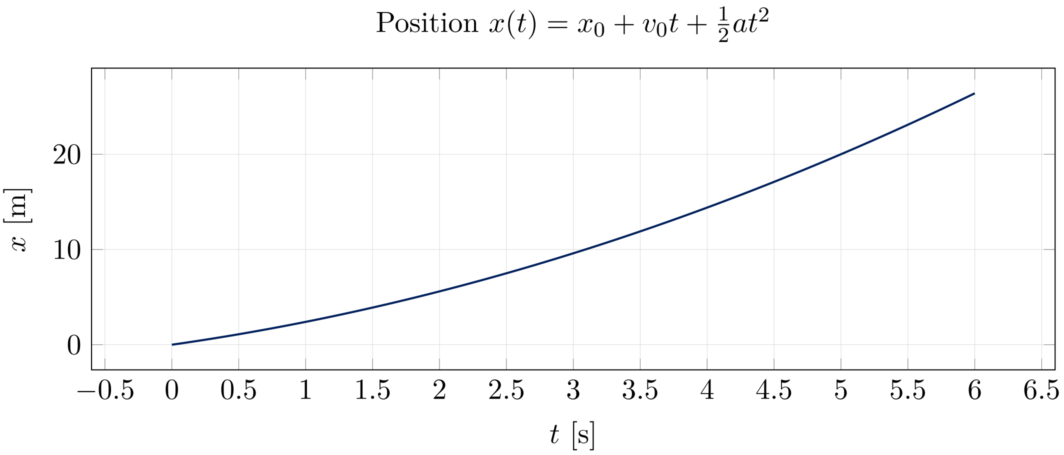


Figure 1.1: Position for constant acceleration $a = 0.8\,\mathrm{m/s^2}$, $x_0 = 0$, $v_0 = 2\,\mathrm{m/s}$.

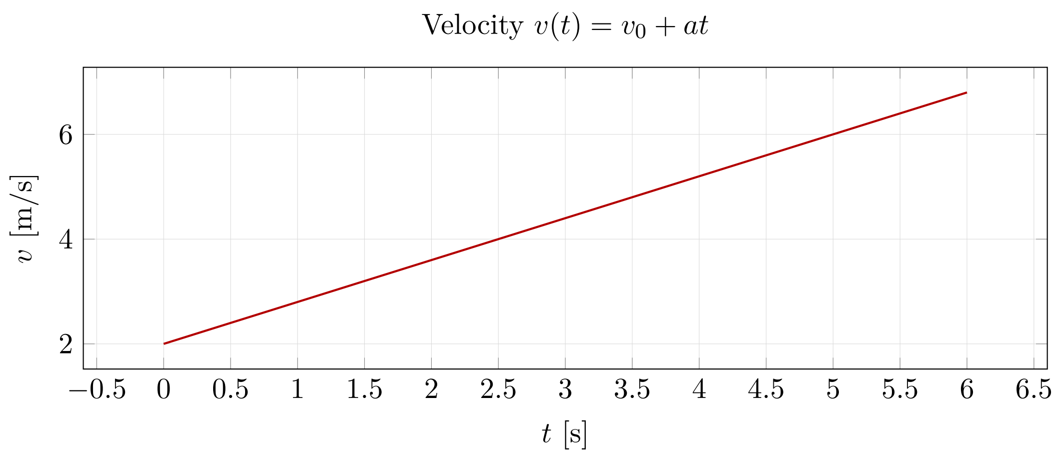


Figure 1.2: Velocity grows linearly with time under constant acceleration.

Worked Example: Elevator Ride

An elevator starts from rest, accelerates at $1.0\,\mathrm{m/s^2}$ for $1.5\,\mathrm{s}$, cruises at $1.5\,\mathrm{m/s}$ for $2.0\,\mathrm{s}$, then decelerates at $-1.0\,\mathrm{m/s^2}$ for $1.5\,\mathrm{s}$ to a stop. The total displacement equals the area under $v(t)$: $s = \frac{1}{2}(1.5)(1.5) + (1.5)(2.0) + \frac{1}{2}(1.5)(1.5) = 5.25\,\mathrm{m}$. Figure 1.3 sketches the piecewise velocity and shades the area equal to displacement.

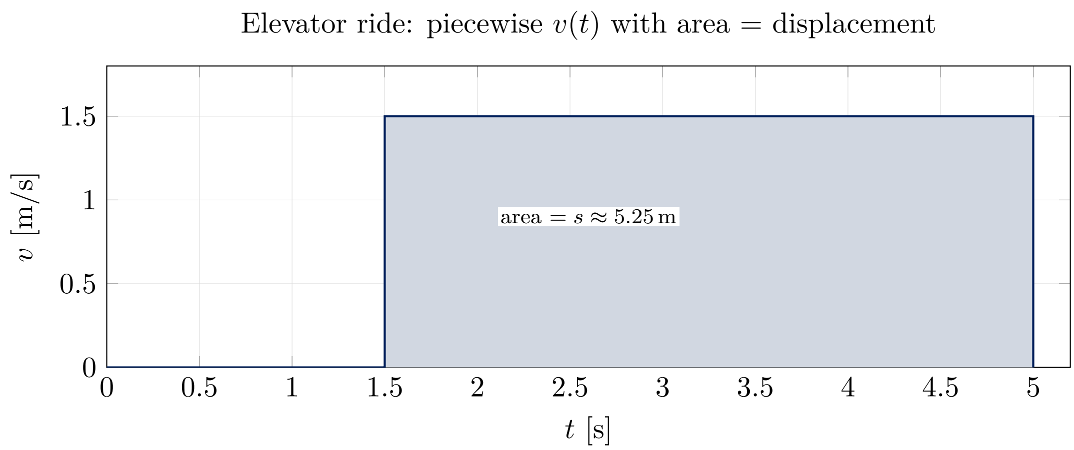


Figure 1.3: Piecewise velocity for a short elevator ride; displacement equals the shaded area under $v(t)$.

1.3 Units, Dimensions, and Orders of Magnitude

We adopt the International System of Units (SI) throughout and routinely check dimensional consistency. Typical scales for everyday objects are shown in Figure 1.4. Developing a sense for scale helps you sanity-check results: if a car ”accelerates” at $50\,\mathrm{m/s^2}$ for ten seconds, your estimate should raise an eyebrow.

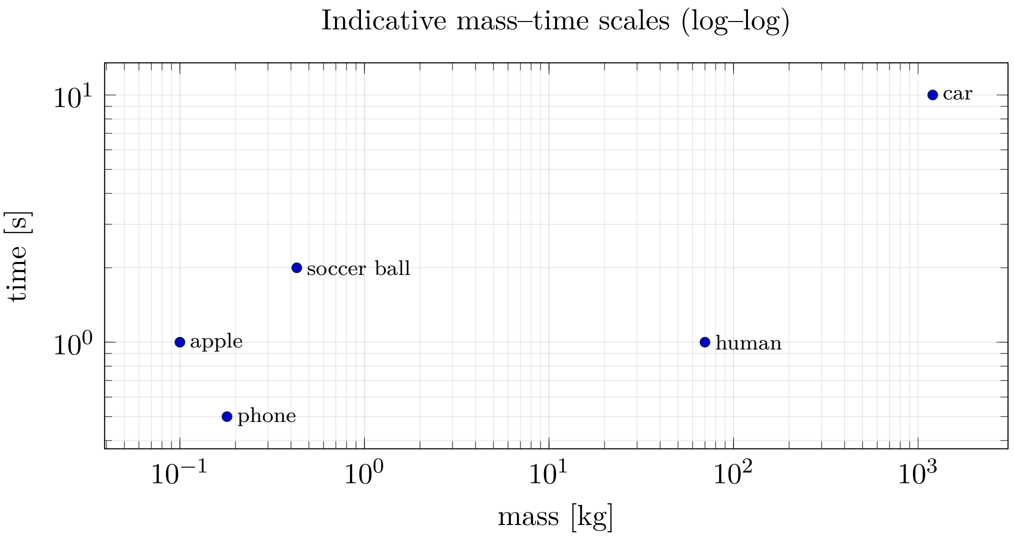


Figure 1.4: Everyday mass and time scales span orders of magnitude.

One-Line Estimate: Train vs. Light

At $v = 50\,\mathrm{m/s}$, the ratio to light speed is $v/c \approx 1.7 \times 10^{-7}$—decisively non-relativistic.

Quick Drop: Phone from Pocket

From $h_0 = 1.2\,\mathrm{m}$, the ideal fall time is $T = \sqrt{2h_0/g} \approx 0.49\,\mathrm{s}$ and the impact speed $|v| \approx gT \approx 4.9\,\mathrm{m/s}$. Air drag is small over this height but does reduce speed slightly.

Dimensional Check

If $s = \frac{1}{2}at^2$ describes displacement under constant acceleration, then $[s] = \mathrm{L}$ and $[at^2] = \mathrm{L\,T^{-2}\,T^2} = \mathrm{L}$—consistent.

1.4 The Structure of Physical Theories

Mechanics organizes knowledge as: states, laws, and solutions. A *state* summarizes what matters now (positions and velocities). *Laws* tell how the state changes (forces produce acceleration). With *initial conditions*, we compute a *solution*—a trajectory through time.

Theorem 1 (Determinism, Informal)**.** *For a well-posed system of ordinary differential equations with smooth right-hand side, there exists a unique solution through each initial state for some time interval.*

In practice, most Newtonian models are initial-value problems (IVPs): given $(x(0), v(0))$ and a force law $\boldsymbol{F}(x, v, t)$ that is smooth (or at least Lipschitz) in its arguments, the motion is locally unique. When forces change discontinuously (e.g., impacts or switching friction regimes), we treat those moments as events and restart the IVP with updated conditions.

Analogy: GPS Trail vs. Rubber Band

A GPS trail is a recorded trajectory; Newton's laws act like a stretched rubber band that pulls the state forward according to the forces—given the same start, you trace the same path.

1.5 Math vs. Physics in This Book

The main text favors intuition and application; short formal notes supply precise definitions and statements when helpful. Use both: intuition to guide, and math to verify. When you feel lost, scan the callouts and figures first, then read the surrounding text for the logic that binds them.

Try in 60 seconds

Quick, confidence-building tasks:

- Point at three objects and say whether Newtonian mechanics applies and why.
- Write one quantity with units from daily life (e.g., speed in m/s) and convert it.
- In one sentence, define "state, law, solution" for the falling-ball story.

1.6 Exercises

Try a few light, insight-building tasks.

1. **Newtonian or not?** For each scenario decide if Newtonian mechanics is appropriate and justify in one sentence: a passenger jet at 900 km/h; an electron in a chip; a satellite in low Earth orbit.
2. **Back-of-the-envelope.** A commuter train at 50 m/s: estimate v/c and comment on whether relativistic effects matter. Repeat for a racing car at 100 m/s.
3. **Dimensional check.** Which is dimensionally consistent for force: (a) $F = mv$, (b) $F = ma$, (c) $F = ma^2$? Explain using unit symbols.
4. **Practical: Elevator ride.** Time a start–stop elevator segment with your phone. Sketch a rough $v(t)$ based on sound/feel; mark where acceleration is positive/negative.
5. **Practical: Everyday scales.** List three objects (fruit, backpack, bicycle) and guess their masses and typical timescales for an action (falling, lifting, rolling). Place your guesses on a hand-drawn log–log mass–time plot similar to Figure 1.4.

1.7 Summary and Review

Quick checklist of what you should now recognize:

- The Newtonian regime and when it breaks (relativity/quantum).
- Core quantities: position, velocity, acceleration, force, energy, momentum.
- Dimensional analysis as a guardrail for sanity checks.
- The structure *state + laws + initial data* $\Rightarrow$ *solution*.
- Figures as arguments: slope $\leftrightarrow$ velocity; area $\leftrightarrow$ accumulation (preview).

1.8 Where We're Heading Next

In Chapter 2 we develop the mathematical language behind the pictures: functions and their graphs, vectors, derivatives and integrals, and ordinary differential equations. This vocabulary lets us turn words like "speeding up" into equations we can analyze and, later, solve.

Common Pitfalls

Short reminders:

- Mixing units (e.g., km/h with m/s) without converting.
- Forgetting to define a clear system, axis, and positive direction.
- Reading values instead of slopes when interpreting change.

Try in 60 seconds

Tiny tasks:

- Convert 36 km/h to m/s.
- Declare a positive direction and state it aloud on a quick sketch.
- Point to a graph and say what the slope means physically.

Chapter 2

Mathematical Language of Newtonian Physics

This chapter builds the math we use throughout the book—functions, vectors, derivatives, integrals, and ordinary differential equations (ODEs). We keep it intuitive and visual, with analogies and small figures to anchor each idea. The goal is not symbol-pushing for its own sake, but a compact language that makes motion problems easy to set up and reason about. If you are new to some concepts, do not worry: each one will be motivated by a physical picture first and formalized second.

Learning Objectives

By the end, you can read a position–time graph for slope and area, compute basic vector operations, take simple derivatives and integrals in motion contexts, and recognize an ODE and how a step-by-step method approximates its solution.

Symbols at a Glance

Quick legend for this chapter:

- $\boldsymbol{r}(t)$ position vector; $\boldsymbol{v} = \dot{\boldsymbol{r}}$ velocity; $\boldsymbol{a} = \dot{\boldsymbol{v}}$ acceleration
- $x(t)$, $v(t)$ scalars for 1D examples; t time
- $\cdot$ dot product; $\times$ cross product (3D)

Analogy: Motion Toolkit

Think of this chapter as a compact toolkit: graphs for seeing change, arrows (vectors) for direction and size, calculus for turning slopes into rates and areas into totals, and ODEs for encoding "laws cause change."

Roadmap

We move from graphs of motion (Figures 2.1 and 2.2) to vectors and their operations (Figure 2.4), then to differentiation/integration as change and accumulation (Figure 2.5). Finally, we preview ODEs and a simple numerical scheme via a schematic comparison (Figure 2.7).

2.1 Functions and Graphs

Functions are rules that take an input and produce an output. In mechanics, the most common function is a trajectory $x(t)$: position as a function of time. When the function is smooth, the graph carries two built-in "tools": the slope of the curve (velocity) and how fast that slope changes (acceleration). Reading a graph is thus a mechanical skill: eyes for slope, eyes for area, eyes for curvature.

Analogy: Storyline

A function is a storyline. Time t indexes the pages; the value (e.g., x) tells you where the character is on each page. A steady slope is a calm walk; a changing slope is a sprint or a sudden stop.

As shown in Figure 2.1, a smooth $x(t)$ lets us talk about slopes (instantaneous velocity) and curvature (how acceleration feels). The corresponding velocity function $v(t)$ appears in Figure 2.2.

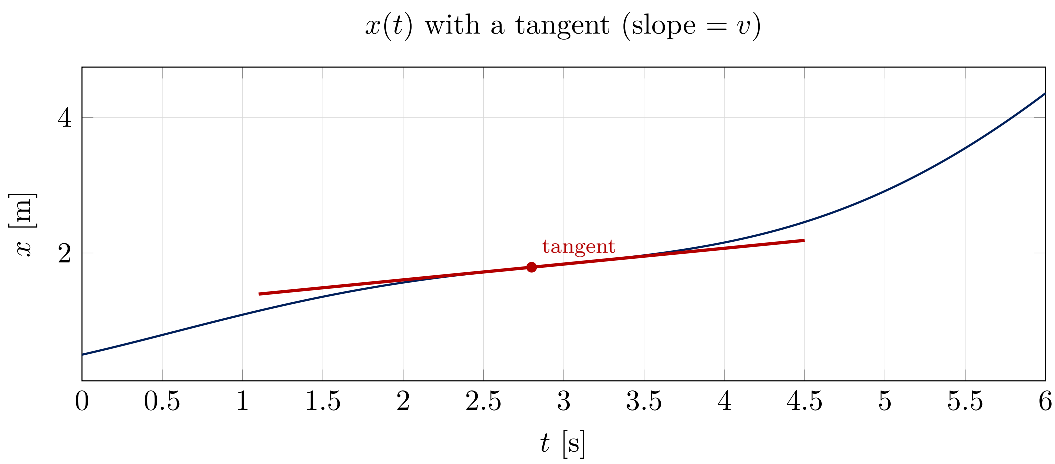


Figure 2.1: A position–time graph with a tangent: the slope gives instantaneous velocity.

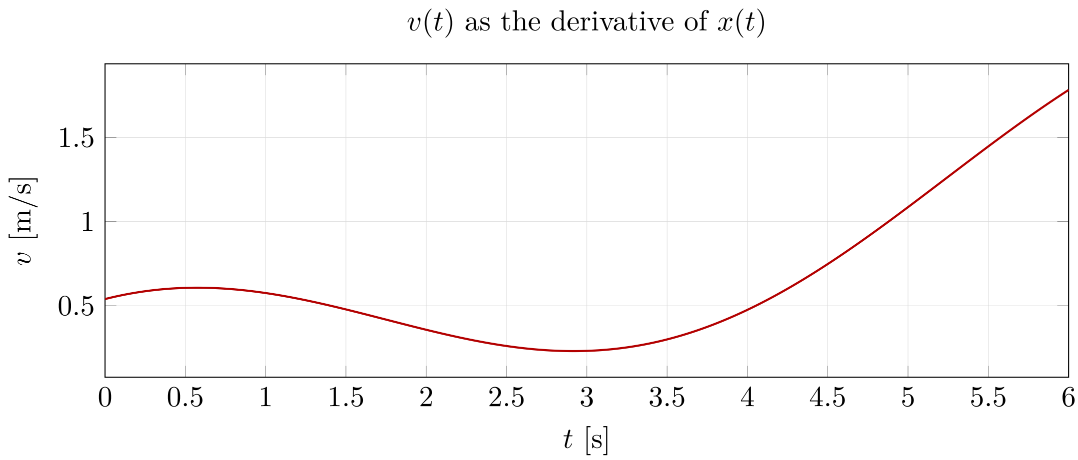


Figure 2.2: Velocity $v(t)$ is the derivative (slope) of $x(t)$.

Worked Example: River Crossing

A boat heads due east at $v_{\text{boat}} = 2.0\,\text{m/s}$ across a river with current $v_{\text{current}} = 1.0\,\text{m/s}$ due north. The ground-frame velocity is the vector sum with magnitude $|\boldsymbol{v}| = \sqrt{2.0^2 + 1.0^2} \approx 2.24\,\text{m/s}$ at angle $\theta = \tan^{-1}(1/2) \approx 26.6°$ north of east. Figure 2.3 draws the component arrows and the resultant in the ground frame.

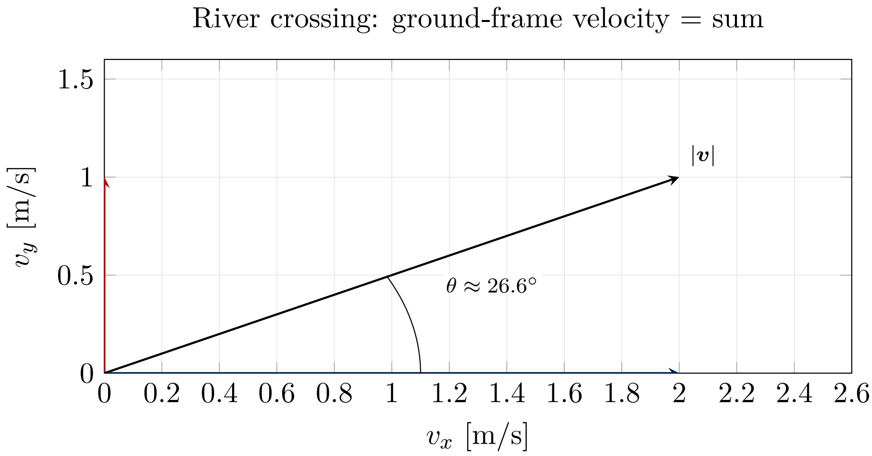


Figure 2.3: Velocity components add head-to-tail; the current deflects the boat downstream.

2.2 Vectors in 2D and 3D

Vectors capture magnitude and direction. In two dimensions we write $\boldsymbol{r} = (x, y)$; in three, $\boldsymbol{r} = (x, y, z)$. Unit vectors $\hat{\boldsymbol{\imath}}, \hat{\boldsymbol{\jmath}}, \hat{\boldsymbol{k}}$ point along the coordinate axes and let us decompose motion into simple, independent pieces. A good habit: sketch arrows, label their components, and check that magnitudes and directions make sense before computing. The dot product encodes alignment ($\boldsymbol{a} \cdot \boldsymbol{b} = \|\boldsymbol{a}\|\,\|\boldsymbol{b}\| \cos\theta$); in 3D the cross product captures oriented area ($\|\boldsymbol{a} \times \boldsymbol{b}\| = \|\boldsymbol{a}\|\,\|\boldsymbol{b}\| \sin\theta$, right-hand rule for direction). Checking units after vector operations is as important as in scalar algebra.

Analogy: Shopping List + Arrows

Components (x, y) are like a two-item shopping list—“east” and “north” steps that add up to the full trip. The arrow picture is the map view of the same list. The dot product measures how much two trips agree, while the cross product (in 3D) measures the area your trips would sweep out together.

Figure 2.4 shows vector addition via the parallelogram rule and highlights the dot product angle.

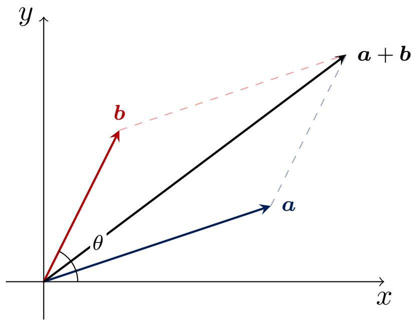


Figure 2.4: Vector addition via the parallelogram rule; angle θ appears in the dot product $\boldsymbol{a} \cdot \boldsymbol{b} = \|\boldsymbol{a}\| \, \|\boldsymbol{b}\| \cos\theta$.

2.3 Differentiation and Integration of Motion

Differentiation measures instantaneous change; integration measures accumulated effect. If velocity is the rate at which position changes, then position is the accumulated effect of velocity. The two are inverse operations when everything is smooth. When motion is piecewise smooth (e.g., sudden braking), the graph still tells the story: slopes change abruptly, and areas still add up.

Analogy: Speedometer and Odometer

The derivative is your speedometer (instant reading); the integral is your odometer (total distance).

Suppose acceleration is a square pulse in time. As previewed in Figure 2.5, the area under $a(t)$ adds to velocity $v(t)$. This is the graphical way to remember the fundamental theorem of calculus in a mechanics setting.

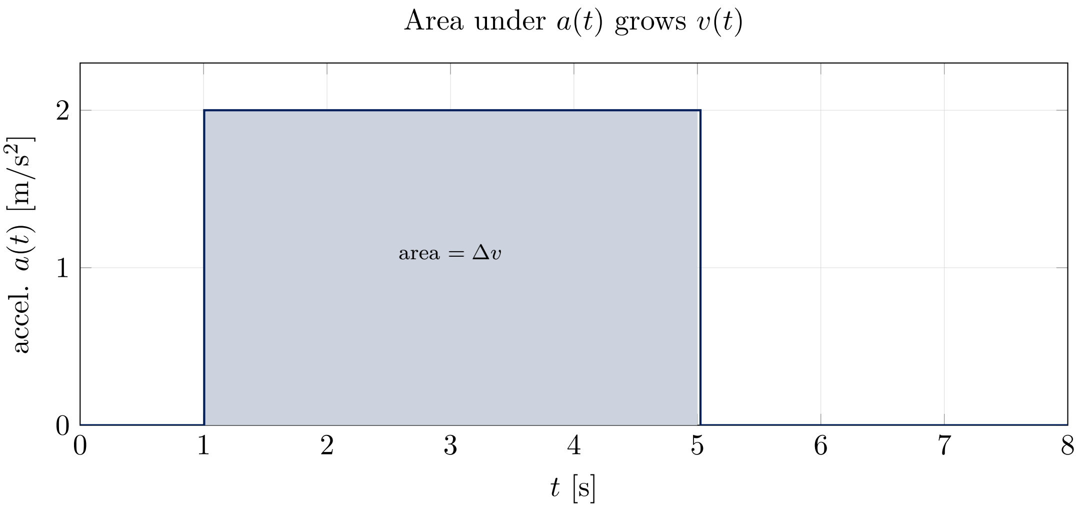


Figure 2.5: Acceleration as a pulse: the shaded area equals the increase in velocity.

Worked Example: Braking Distance

A car slows uniformly from 25 m/s to 0 in 4.0 s. The velocity graph is a straight line, and the stopping distance equals the triangular area: $s = \frac{1}{2}(25)(4.0) = 50\,\text{m}$. The constant acceleration is $a = \Delta v/\Delta t = -6.25\,\text{m/s}^2$. Figure 2.6 shows the linear $v(t)$ and shaded area.

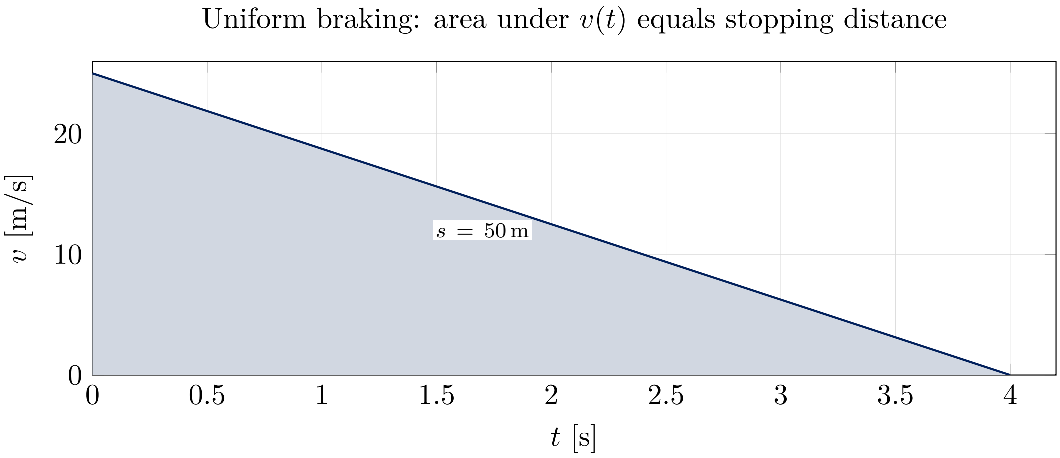


Figure 2.6: Linear $v(t)$ during braking; the shaded triangle's area is the stopping distance.

Integral in Practice: Flow Total

A faucet runs with rate $q(t) = 0.4 + 0.1\,\sin(\pi t/5)$ L/s over $0 \le t \le 10$ s. The total volume is $V = \int_0^{10} q(t)\,dt = 4.0\,\text{L}$ (the sine contributes zero over a full period).

2.4 Ordinary Differential Equations in Mechanics

An ordinary differential equation (ODE) relates a function to its derivatives. In mechanics, Newton's second law $m\,\ddot{x} = F(x, \dot{x}, t)$ is an ODE for motion in one dimension. An initial value problem (IVP) specifies $x(0)$ and $\dot{x}(0)$ and asks for the future.

Analogy: Recipe + First Bite

The ODE is the recipe; the IVP adds the first bite (initial state) so you know exactly how the taste evolves.

For constant acceleration a, the IVP solution is $x(t) = x_0 + v_0 t + \frac{1}{2}at^2$ and $v(t) = v_0 + at$—consistent with Figure 2.2. When forces depend on position or velocity, the ODE encodes feedback (e.g., drag slows you more when you go faster).

2.5 Numerical Approximation of Motion (Preview)

When we cannot solve an ODE exactly, we approximate it over small time steps. The Euler method updates

$$v_{n+1} = v_n + a(t_n, x_n, v_n)\,\Delta t,$$
$$x_{n+1} = x_n + v_n\,\Delta t.$$

As illustrated schematically in Figure 2.7, the numerical path follows the exact curve in small forward steps. Smaller steps improve accuracy but cost time; too-large steps can wander off the true solution. Later we will compare simple schemes to better ones and discuss stability (numerical solutions that behave physically even for larger steps).

One Euler Step: Linear Drag

For $\dot{v} = g - (c/m)\,v$ with $g = 9.81\,\mathrm{m/s^2}$ and $c/m = 0.60\,\mathrm{s^{-1}}$, take $v_0 = 0$ and $\Delta t = 0.2\,\mathrm{s}$. One forward step gives $v_1 \approx v_0 + \Delta t\,[g - (c/m)v_0] = 1.962\,\mathrm{m/s}$. Smaller Δt tracks the curve better.

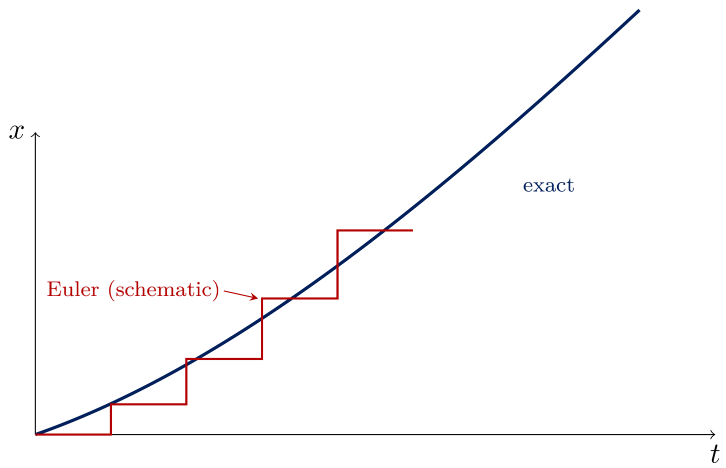


Figure 2.7: Analytic curve (smooth) versus forward Euler steps (staircase). The staircase advances in small forward steps along the time axis, approximating the smooth motion.

2.6 Snell's Law from Least Time (Lifeguard Analogy)

A lifeguard runs faster on sand than in water and wants to reach a swimmer as fast as possible. The fastest path is not a straight line—it bends at the shoreline so that

$$\frac{\sin\theta_1}{v_1} = \frac{\sin\theta_2}{v_2},$$

where v_1 and v_2 are speeds on sand and in water, and angles are measured from the normal. Defining a refractive index proportional to $1/v$ recovers *Snell's law* $n_1 \sin\theta_1 = n_2 \sin\theta_2$.

Before Figure 2.8, picture two straight segments meeting the shoreline at one point; we optimize that meeting point to minimize total time. The derivative step mirrors those in Appendix A.

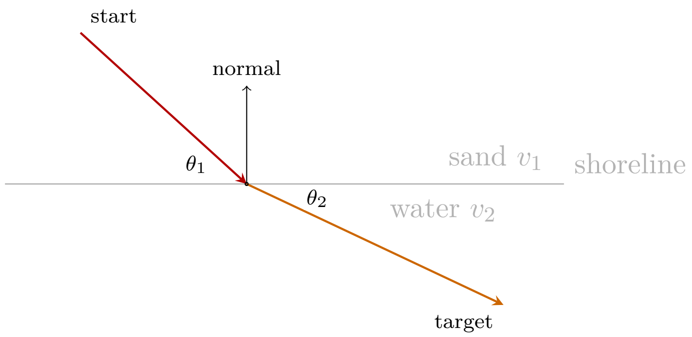


Figure 2.8: Fastest two-segment path across media with speeds v_1 (sand) and v_2 (water). Equalizing $\dfrac{\sin\theta}{v}$ across the boundary minimizes time and yields Snell's law.

One-Line Derivation

Let the shoreline point be x along the beach, with distances $d_1(x)$ on sand and $d_2(x)$ in water. Time $T(x) = d_1(x)/v_1 + d_2(x)/v_2$. Setting $T'(x) = 0$ gives $\frac{\sin\theta_1}{v_1} = \frac{\sin\theta_2}{v_2}$. Replace $1/v$ by refractive index n to obtain $n_1 \sin\theta_1 = n_2 \sin\theta_2$. See Appendix A for the derivative idea.

2.7 Exercises

Short, friendly tasks to practice the math language.

1. **Read a graph.** For $x(t) = 1 + 0.5t + 0.1t^2$, compute $v(t)$ and $a(t)$. At $t = 2$, estimate the slope from a sketch and compare with the formula.
2. **Vector playground.** Take $\boldsymbol{a} = (2, 1)$ and $\boldsymbol{b} = (1, 2)$. Compute $\boldsymbol{a}+\boldsymbol{b}$, $\boldsymbol{a}\cdot\boldsymbol{b}$, and the angle θ between them. Sketch the parallelogram as in Figure 2.4.
3. **Practical: Two-axis walk.** Walk 20 steps east, then 15 steps north. Mark your start and end on graph paper. Measure the displacement vector and its magnitude.
4. **Area and change.** Suppose $a(t) = 0$ except $a = 2\,\mathrm{m/s^2}$ from $t = 1$ to $t = 3$. If $v(0) = 0$, what is $v(4)$? Use area under $a(t)$ as in Figure 2.5.
5. **One Euler step.** For free fall without air $\ddot{x} = -g$, take $g = 9.8$, $\Delta t = 0.2$, $x_0 = 0$, $v_0 = 0$. Do two Euler updates by hand and compare $x(0.4)$ with the exact $\frac{1}{2}gt^2$.

2.8 Summary and Review

Checklist of ideas and tools:

- Functions and their graphs encode motion; slopes and areas have direct physical meaning.
- Vectors represent magnitude and direction; addition and dot product capture geometry and work.
- Differentiation measures change; integration accumulates effects.
- Newtonian motion fits naturally as ODEs with initial data.
- Simple numerical methods (Euler) approximate motion step by step.

2.9 Where We're Heading Next

In Chapter 3 we specialize to one-dimensional kinematics. We will relate $x(t)$, $v(t)$, and $a(t)$ in detail, master constant-acceleration motion, and interpret slopes/areas on data-like plots—building on the language you developed here.

Common Pitfalls

Avoid these slips:

- Confusing slope (rate) with value when reading graphs.
- Dropping vector arrows and mixing magnitudes with components.
- Forgetting that areas under $v(t)$ give displacement (signed), not always distance.

Try in 60 seconds

Tiny wins to cement the language:

- Sketch any increasing function and draw a tangent; label its slope.
- Add two arrows head-to-tail and name the parallelogram.
- Take one forward Euler step for $x' = 2$ with $x(0) = 0$ and $\Delta t = 0.3$.

Part II

One-Dimensional Motion

Part II Overview

This part makes the toolkit move along a line. Chapter 3 studies position, velocity, and acceleration in one dimension using slope/area reasoning and constant-acceleration models. Chapter 4 introduces forces in 1D, connects them with Newton's laws, and uses simple numerical updates to compare predictions with analytic motion.

Chapter 3

Kinematics in One Dimension

Motion along a straight line is the laboratory where we learn to read graphs and translate between position $x(t)$, velocity $v(t)$, and acceleration $a(t)$. We emphasize pictures, consistent sign conventions, and short formulas that say exactly what the pictures say. Our aim is fluency: seeing a graph and immediately knowing what its slope means physically; seeing a formula and picturing its curve.

Learning Objectives

You will interpret $x(t)$, $v(t)$, and $a(t)$ graphs, carry out constant-acceleration calculations, and tell coherent "slope/area stories" for everyday 1D motion.

Symbols at a Glance

Quick legend used throughout this chapter:

- $x(t)$ position along a line; Δx displacement between two times
- $v(t) = \dot{x}(t)$ velocity (slope of x); $a(t) = \dot{v}(t)$ acceleration (slope of v)
- t time; areas under $v(t)$ give displacement

Analogy: One Track, Three Cameras

Think of a toy car on a straight track. Three synchronized cameras watch the same action: the x-*camera* records position, the v-*camera* records the slope of $x(t)$, and the a-*camera* records how the slope itself changes. Reading kinematics is learning to jump between these feeds.

3.1 Position, Displacement, Velocity, Acceleration

Choose a coordinate axis, a positive direction, and a reference point $x = 0$. Displacement is change in position $\Delta x = x_2 - x_1$ (a *signed* quantity), whereas distance traveled is always non-negative. Average velocity is $\Delta x/\Delta t$; instantaneous velocity is the slope $v(t) = \dot{x}(t)$. Acceleration is the slope of velocity, $a(t) = \dot{v}(t) = \ddot{x}(t)$. A handy mental model is the car dashboard: the trip counter estimates average velocity over an interval; the speedometer reads the instantaneous velocity; how "hard" you feel pressed into the seat hints at acceleration.

Sign Convention

Choose a positive direction and stick to it. With that fixed, a negative velocity means moving opposite that direction; a negative acceleration means the velocity is decreasing in that direction.

In Figures 3.1 to 3.3 we show a consistent triplet: a simple motion with steadily increasing velocity (constant acceleration).

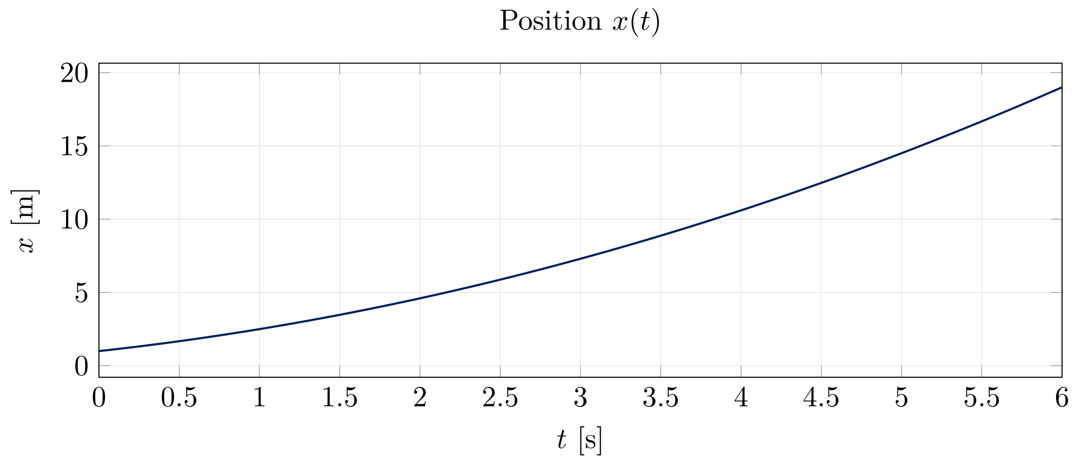


Figure 3.1: A sample position curve with constant acceleration.

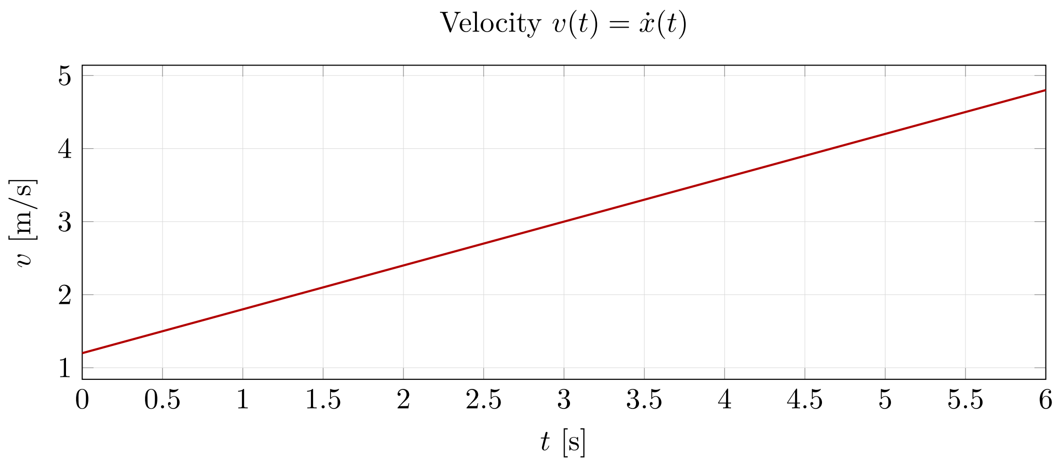


Figure 3.2: Velocity grows linearly when acceleration is constant.

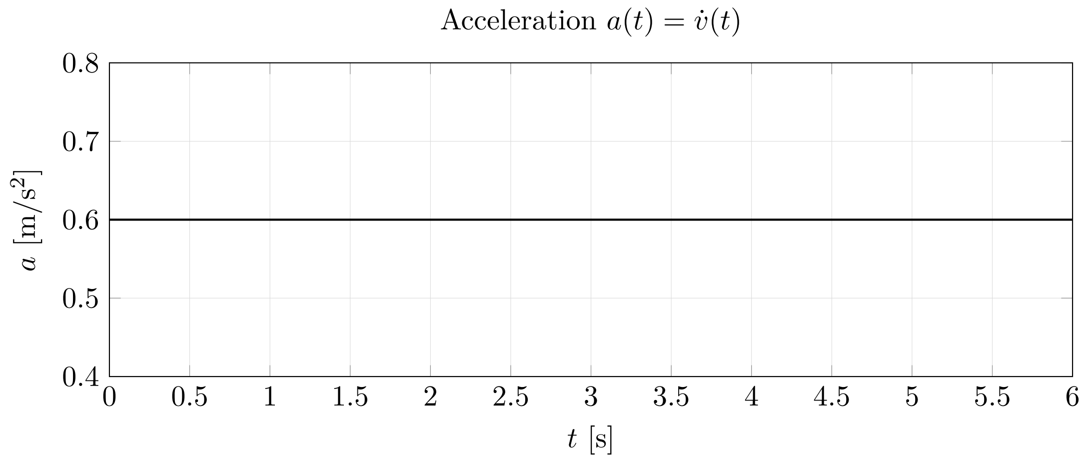


Figure 3.3: Constant acceleration as a horizontal line.

3.2 Graphical Kinematics

When you face a graph, ask two questions: "What is the slope here?" and "What is the area between these times?" Slope translates to an instantaneous rate (e.g., velocity from $x(t)$), and area translates to an accumulated effect (e.g., displacement from $v(t)$). Remember that areas under $v(t)$ are *signed*: if v dips below zero, the negative area reduces the net displacement. Two conversions matter most:

- Slope of $x(t) \Rightarrow v(t)$; slope of $v(t) \Rightarrow a(t)$.
- Area under $v(t)$ between t_1 and $t_2 \Rightarrow$ displacement $\int_{t_1}^{t_2} v(t)\,\mathrm{d}t$.

As previewed in Figure 3.4, the shaded area under a linearly increasing velocity gives the change in position. Picture laying down thin rectangular tiles under the curve between t_1 and t_2—their combined area equals the displacement.

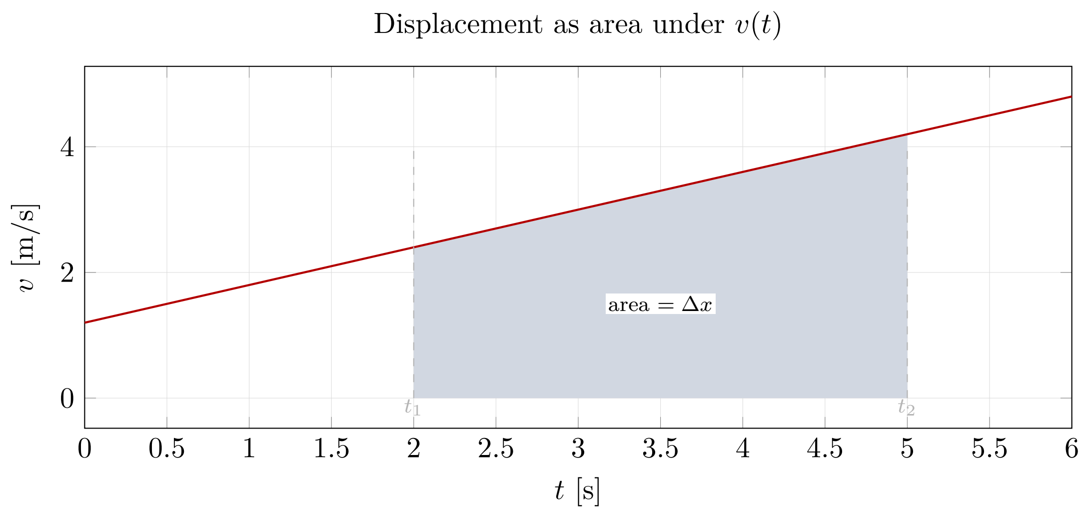


Figure 3.4: Area under $v(t)$ between two times equals displacement.

3.3 Uniform and Non-Uniform Motion

Uniform motion has constant velocity ($a = 0$); non-uniform motion has $a \neq 0$. In uniform motion, $x(t)$ is a straight line; in uniformly accelerated motion, $v(t)$ is a straight line. For constant acceleration a, the classic relations are

$$v(t) = v_0 + at, \qquad x(t) = x_0 + v_0 t + \tfrac{1}{2}at^2, \qquad v^2 = v_0^2 + 2a(x - x_0).$$

Each is a compact way to say what the graphs already show in Figures 3.1 to 3.3.

3.4 Real-World Examples

Consider a car that accelerates smoothly from rest, cruises, then brakes to a stop. The $v(t)$ sketch looks like a hill: up, flat, down. The displacement is the total area under that hill. If your daily commute has two such hills (stop-and-go), the day's distance is the sum of the two areas. A moving walkway and your walking speed add linearly in $v(t)$—a nice reminder that the graph is a physical story.

Worked Example: 0–10 m Sprint

A sprinter starts from rest, accelerates at $a = 4.0\,\mathrm{m/s^2}$ up to $v = 8.0\,\mathrm{m/s}$, then holds that speed. The distance to 10 m is the area under $v(t)$. During acceleration: $s_1 = \frac{1}{2}at_1^2$ with $t_1 = \frac{v}{a} = 2.0\,\mathrm{s}$, so $s_1 = 8.0\,\mathrm{m}$. The remaining 2.0 m at 8.0 m/s takes $t_2 = 0.25\,\mathrm{s}$. Thus the 10 m time is $T = t_1 + t_2 = 2.25\,\mathrm{s}$. Figure 3.5 shows the piecewise $v(t)$ with the area to $t = T$ shaded.

Piecewise sprint velocity to 10 m

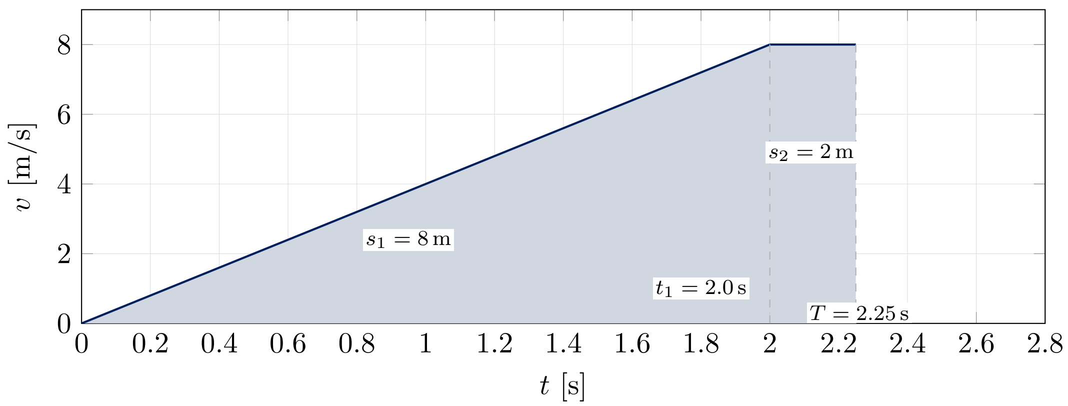


Figure 3.5: A simple sprint model: accelerate linearly to 8 m/s, then hold speed until the 10 m area is reached.

3.5 Exercises

Practice translating words to graphs and graphs to formulas.

1. **Slope and area.** From the line $v(t) = 3 + 0.5t$, find $x(t)$ with $x(0) = 0$. Mark the area on a sketch for $t \in [0, 4]$ and compare with your formula.

2. **Units and signs.** A jogger runs west and slows down: v is negative and a is positive/negative? Explain.

3. **Constant acceleration.** With $a =$ const, derive $v^2 = v_0^2 + 2a(x - x_0)$ from $v\,\mathrm{d}v = a\,\mathrm{d}x$.

4. **Practical: Escalator timing.** Time a ride up an escalator while walking. Sketch $v(t)$ for "stand" versus "walk" and compare areas (displacements).

5. **Practical: Bike start/stop.** From rest, pedal to cruising speed, then brake gently. Have a friend film a wheel marker and estimate $v(t)$ from frame spacing; identify where $a(t)$ changes sign.

3.6 Summary and Review

A quick checklist of ideas to keep on one page:

- $x(t)$, $v(t)$, $a(t)$ form a triangle of ideas: slopes go down the chain, areas climb up.
- Constant acceleration yields straight $v(t)$ and parabolic $x(t)$.
- Displacement is area under $v(t)$; sign conventions make predictions consistent.

3.7 Where We're Heading Next

In Chapter 4 we introduce forces and Newton's laws in one dimension, connect $F = ma$ to the $v(t)$ and $x(t)$ stories from this chapter, and compare analytic solutions with simple numerical updates.

Common Pitfalls

Quick cautions:

- Reading heights instead of slopes when interpreting $x(t)$ and $v(t)$.
- Mixing displacement (signed) with distance (always positive).
- Switching sign conventions mid-problem.

Try in 60 seconds

Fast checks:

- Draw any $v(t)$ that makes $x(t)$ stay constant.
- Sketch a $v(t)$ that would make $x(t)$ concave up.
- Read the area under a triangular $v(t)$ between two times.

Chapter 4

Dynamics in One Dimension

In Chapter 3 we learned to read motion; now we learn to *cause* it. Dynamics links pushes and pulls (forces) to changes in motion (accelerations) via Newton's laws. In one dimension the story is crisp: pick an axis, assign signs, add forces along that axis, then use $F_{\text{net}} = ma$ to connect to $v(t)$ and $x(t)$.

Learning Objectives

You will draw clean free-body diagrams, write down $F_{\text{net}} = ma$ with signs, predict qualitative $v(t)$ and $x(t)$ shapes from forces, and perform a first numerical step for a simple 1D model.

Symbols at a Glance

We use m mass, F force, a acceleration, v velocity, x position, c drag coefficient, k spring constant, g gravitational acceleration.

Forces Explain Changes

Kinematics tells you *what* the motion looks like; dynamics tells you *why* it changes. A constant velocity needs no force; a change in velocity betrays a nonzero net force.

4.1 Newton's Laws in 1D

Read the laws with your sign convention in mind and keep everything along your chosen axis:

- First law (inertia): if $F_{\text{net}} = 0$, then v is constant (including the case $v = 0$).
- Second law: $F_{\text{net}} = ma$ with $a = \dot{v} = \ddot{x}$; signs follow your axis choice.
- Third law: forces between interacting bodies come in equal-and-opposite pairs. These act on *different* bodies, so they never cancel within a single free-body diagram.

We keep track of signs with a consistent axis. If "to the right" is positive, then a leftward push is a negative force. A simple mental check: if the net force arrow points right, acceleration should be positive.

Sign Convention

Pick an axis and declare its positive direction at the start of a problem. All forces and accelerations are signed according to that choice—no switching mid-stream.

Analogy: Budget of Pushes

Think of forces like a budget: positive forces are income, negative forces are expenses. The "net force" is your balance; acceleration is how quickly your "motion account" changes.

4.2 Mass and Common Forces

Mass measures inertia (resistance to changes in motion). Typical 1D force components include weight $W = mg$ (downward), normal forces from surfaces, friction (static/kinetic), and simple driving forces. In 1D problems we often project a multi-D situation onto an axis along the motion. A compact way to organize thinking is the free-body diagram: isolate the object and draw all forces with signs according to your axis. A minimal example appears in Figure 4.1.

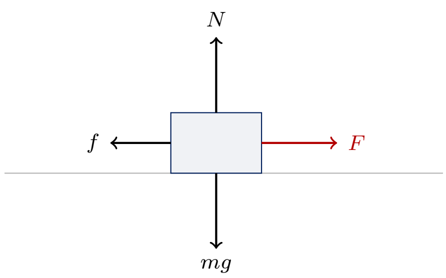


Figure 4.1: Free-body diagram for a block on a horizontal surface with push F and kinetic friction $f = \mu_k N$ opposing the motion.

4.3 Basic 1D Force Models

Three simple models already cover many situations; each makes a different prediction for how velocity should evolve:

- Constant force $F = F_0$ gives $a = F_0/m$ and reproduces the constant-acceleration results from Chapter 3.
- Linear drag $F = -cv$ opposes motion; with a constant drive it produces an exponential approach to a terminal speed $v_{\text{term}} = F_0/c$ with time constant $\tau = m/c$.
- Hooke spring $F = -k(x - x_*)$ pulls toward an equilibrium point x_* (undamped motion is simple harmonic; see Chapter 13).

4.4 From Force to Motion

Once $F_{\text{net}}(x, v, t)$ is specified, Newton's law $m\ddot{x} = F_{\text{net}}$ is an ODE. Sometimes we can solve it exactly; often we can only solve or visualize it approximately. As a case study, Figure 4.2 shows a constant drive opposed by linear drag: the velocity rises quickly at first and then flattens toward a terminal value.

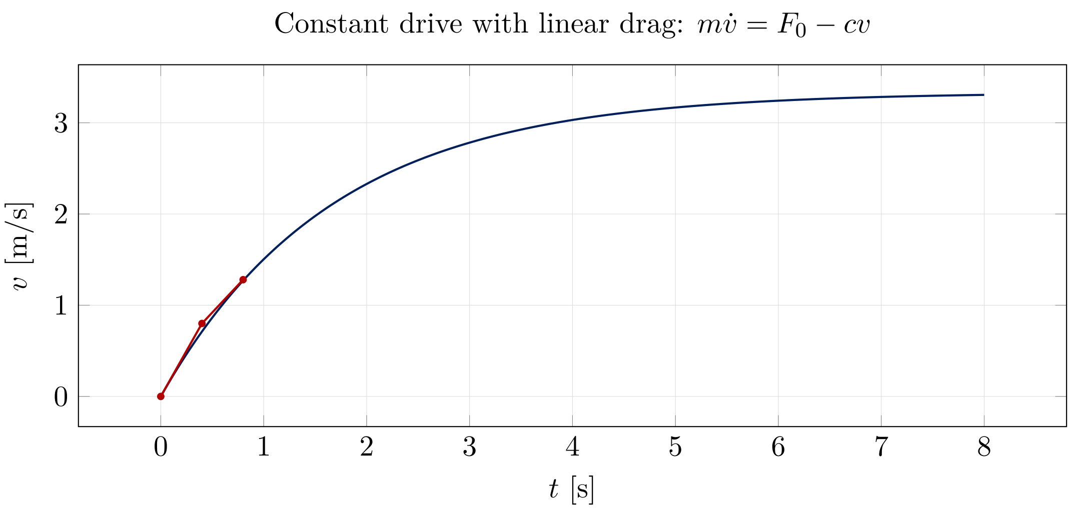


Figure 4.2: Analytic velocity curve (navy) approaching a terminal speed under linear drag, with a few forward Euler markers (red) for illustration. Discussed in Section 4.4.

Worked Example: Coffee Filter Terminal Speed

A lightweight coffee filter falls through air. A simple model is $m\,\dot{v} = mg - cv$ (linear drag). Define the terminal speed $v_T = \frac{mg}{c}$ and the time constant $\tau = \frac{m}{c}$. For $m = 1.5\,\mathrm{g}$ and $c = 0.015\,\mathrm{kg/s}$ one finds $v_T \approx 0.98\,\mathrm{m/s}$ and $\tau \approx 0.10\,\mathrm{s}$. Starting from rest,

$$v(t) = v_T\big(1 - e^{-t/\tau}\big).$$

Figure 4.3 plots $v(t)$ approaching v_T and marks $t = \tau$ (about 63% of v_T).

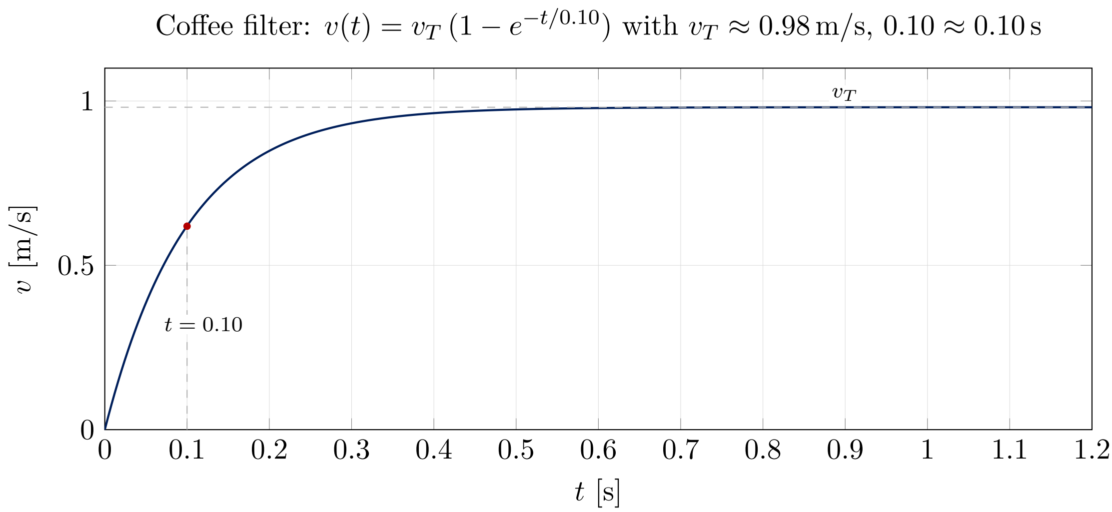


Figure 4.3: Approach to terminal speed under linear drag. After one time constant τ, the speed is about $0.63\,v_T$.

4.5 Numerical Integration in 1D

For $m\,\ddot{x} = F(x, \dot{x}, t)$, one forward Euler step of size Δt reads

$$v_{n+1} = v_n + \tfrac{1}{m} F(x_n, v_n, t_n)\,\Delta t,$$
$$x_{n+1} = x_n + v_n\,\Delta t.$$

Smaller steps improve accuracy; dimensionless groupings (e.g., $c\,\Delta t/m$ in linear drag) help judge stability. A simple pseudocode template is: choose x_0, v_0, then loop updates for $n = 0, 1, 2, \ldots$ with your force model.

4.6 Everyday 1D Dynamics

Many familiar motions are effectively one-dimensional after projection onto an axis. Start with a sentence describing the situation, then sketch the forces and predict $v(t)$ qualitatively before computing.

- Car on a straight road in light traffic: engine push roughly constant at first, air drag increases with speed; $v(t)$ rises toward a plateau (terminal speed).
- Sliding box on a shallow incline: component of weight competes with kinetic friction; if the component wins, $v(t)$ grows linearly; if they match, motion steadies.
- Braking to a stop: roughly constant braking force gives a constant negative acceleration; $v(t)$ is a straight line sloping down to zero and $x(t)$ a concave-down parabola.

4.7 Exercises

Practice going from a free-body diagram to equations of motion, and from force models to qualitative $v(t)$ and $x(t)$ shapes.

1. **Sign conventions.** A cart is pushed left with 3 N while friction exerts 1 N to the right. If right is positive, write F_{net} and the acceleration for $m = 1\,\text{kg}$.
2. **Free-body sketch.** Draw a free-body diagram for a block on a rough horizontal surface pushed to the right; label N, mg, F, and kinetic friction f.
3. **Linear drag.** For $m\dot{v} = F_0 - cv$, show that $v(t) = \frac{F_0}{c}(1 - e^{-ct/m})$ for $v(0) = 0$.
4. **Practical: Pull test.** Gently pull a small object across a table with a rubber band. Note the start (overcoming static friction) versus the sliding (kinetic friction). Sketch the implied F versus t.
5. **One Euler step.** With $m\ddot{x} = F_0 - c\dot{x}$, take $m = 1$, $F_0 = 2$, $c = 0.5$, $\Delta t = 0.2$, $v_0 = x_0 = 0$. Compute v_1, x_1, v_2, x_2.

4.8 Summary and Review

A quick checklist before moving on:

- Newton's three laws specialize cleanly in 1D with signed forces along an axis.
- Common models: constant force, linear drag, and springs already describe many systems.
- From F_{net} to motion: solve $m\ddot{x} = F$ analytically when possible, numerically when needed.

4.9 Where We're Heading Next

In Chapter 5 we lift the same ideas into the plane and space, building free-body diagrams and trajectories that use the vector language introduced in Chapter 2.

Common Pitfalls

Short reminders:

- Missing or inconsistent sign conventions on axes and forces.
- Using static friction where sliding occurs (or vice versa).
- Forgetting that normal forces adjust to constraints and are not always $mg\cos\theta$.

Try in 60 seconds

Micro-tasks:

- Draw a free-body diagram for a pulled sled and label signs along your axis.
- Write $F_{\text{net}} = ma$ for "engine push minus drag".
- Do one Euler step for $m\dot{v} = F_0 - cv$ with $m = 1$, $F_0 = 2$, $c = 0.5$, $v_0 = 0$, $\Delta t = 0.2$.

Part III

Two- and Three-Dimensional Motion

Part III Overview

We now move from lines to planes and space. Chapter 5 develops vector kinematics: representing position, velocity, and acceleration as vectors; projecting motion onto axes; and reading parametric curves with tangent and normal information. Chapter 6 applies these ideas to projectiles and circular motion, building visual intuition for trajectories and centripetal effects.

Chapter 5

Vectors and Kinematics in 2D/3D

In one dimension, arrows could hide; in two and three dimensions, arrows are the language. This chapter develops vector kinematics: we describe position with $\boldsymbol{r}(t)$, velocity with $\boldsymbol{v}(t) = \dot{\boldsymbol{r}}(t)$, and acceleration with $\boldsymbol{a}(t) = \dot{\boldsymbol{v}}(t)$. We keep the narrative visual—sketches first, formulas second. By the end, you should be able to sketch a path, draw tangents and normals confidently, and move between components and geometry with ease.

Learning Objectives

You will represent motion with vectors, project and recombine components, read tangent/normal information from a curve, and connect dot products to alignment and work.

Symbols at a Glance

Quick legend: $\boldsymbol{r}$ position, $\boldsymbol{v}$ velocity, $\boldsymbol{a}$ acceleration, $\hat{\boldsymbol{t}}$ tangent unit, $\hat{\boldsymbol{n}}$ inward normal unit.

Analogy: Map + Breadcrumbs

Imagine a drone leaving breadcrumbs along its path. The vector $\boldsymbol{r}(t)$ is the map pin at time t. The velocity $\boldsymbol{v}(t)$ is the arrow on the breadcrumb showing direction and speed. The acceleration $\boldsymbol{a}(t)$ is the way that arrow itself turns and lengthens.

5.1 Vector Representation of Motion

We write $\boldsymbol{r}(t) = (x(t), y(t))$ in the plane or $(x(t), y(t), z(t))$ in space. Unit vectors $\hat{\boldsymbol{\imath}}, \hat{\boldsymbol{\jmath}}, \hat{\boldsymbol{k}}$ point along the coordinate axes. Differentiation and integration act componentwise, so everything you learned in Chapter 2 carries over with arrows on top.

Two immediate "vector calculus" facts are worth keeping on a sticky note:

- Speed is the magnitude of velocity: $v = \|\boldsymbol{v}\| = \|\dot{\boldsymbol{r}}\|$. The unit tangent is $\hat{\boldsymbol{t}} = \boldsymbol{v}/\|\boldsymbol{v}\|$.
- Arc length s satisfies $\frac{\mathrm{d}s}{\mathrm{d}t} = v$. Using $\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\mathrm{d}s}{\mathrm{d}t}\frac{\mathrm{d}}{\mathrm{d}s} = v\frac{\mathrm{d}}{\mathrm{d}s}$ makes tangent/normal formulas compact.

As shown in Figure 5.1, a parametric curve with a few velocity arrows already tells a story: the red arrows hug the path and tilt as the slope changes. The length of each arrow represents speed; their directions give the tangent.

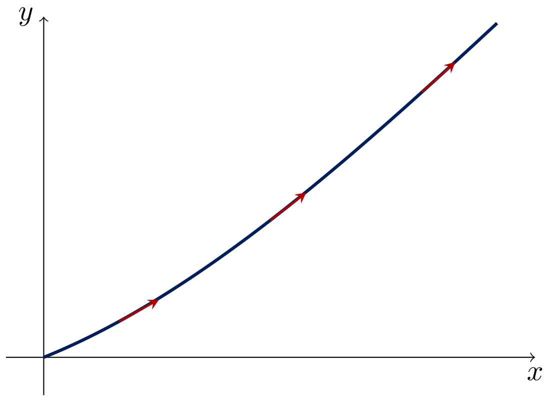


Figure 5.1: A 2D path (navy) with a few tangent velocity arrows (red) indicating direction of motion.

Before we state the reflection rule, it helps to see how any incoming direction at a surface splits into a component parallel to the surface and one along the normal. The geometric split in Figure 5.2 is the picture we will reuse when we derive the vector reflection law in Figure 5.5.

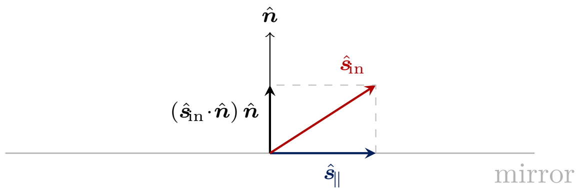


Figure 5.2: Vector decomposition at the mirror: the incoming direction splits into a part parallel to the surface and a part along the normal; reflection flips only the normal part.

Worked Example: Straight Across the River

A canoe can make $v_{\text{boat}} = 2.0\,\text{m/s}$ relative to still water. The river current is $v_{\text{current}} = 0.50\,\text{m/s}$ to the east. To land directly opposite (no downstream drift), steer a little west of north so the boat's horizontal component cancels the current: $v_{\text{boat}} \sin\phi = 0.50$. Thus $\phi = \sin^{-1}(0.50/2.0) \approx 14.5°$. The ground-frame velocity then points straight north with magnitude $\sqrt{2.0^2 - 0.50^2} \approx 1.94\,\text{m/s}$. Figure 5.3 shows the component arrows.

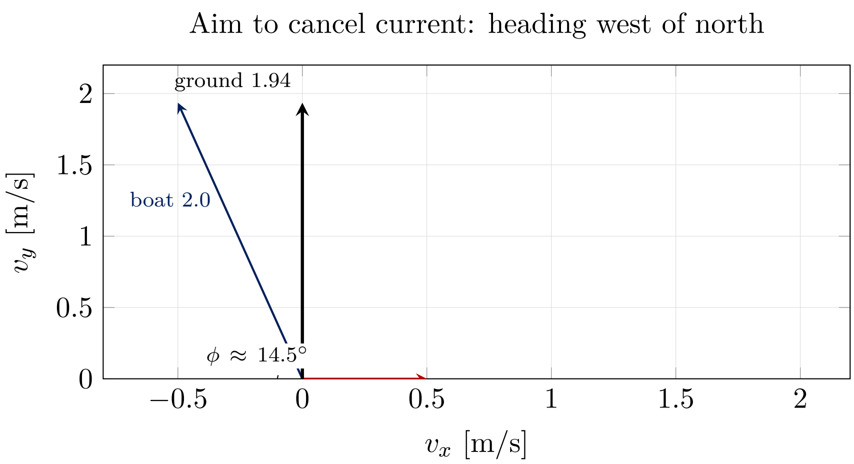


Figure 5.3: Choose a heading so the boat's westward component cancels the eastward current. The ground-frame velocity points straight north.

5.2 Components, Projection, and Geometry

Vectors project naturally onto axes. The dot product $\boldsymbol{a} \cdot \boldsymbol{b} = \|\boldsymbol{a}\| \, \|\boldsymbol{b}\| \cos\theta$ measures alignment; in mechanics it yields work: $W = \int \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r}$. The cross product (in 3D) encodes perpendicularity and areas; it will reappear for angular momentum and torque.

Analogy: Two Narrators

Describing motion with components is like having two narrators—one for east–west, one for north–south. Each tells a simple story; together they tell the whole plot.

To make components concrete, Figure 5.4 shows a vector with its x- and y-projections. Reading this picture is a skill: eyes move horizontally and vertically to match lengths with components before any computation.

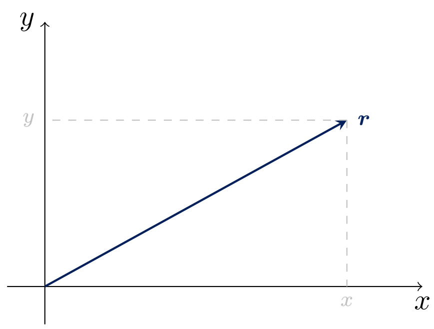


Figure 5.4: A vector $\boldsymbol{r}$ with dashed projections on the coordinate axes.

5.3 Reflection as a Vector Rule

Mirrors are a perfect vector application. At a flat surface with unit normal $\hat{\boldsymbol{n}}$, an incoming unit direction $\hat{\boldsymbol{s}}_{\text{in}}$ reflects to

$$\boxed{\hat{\boldsymbol{s}}_{\text{out}} = \hat{\boldsymbol{s}}_{\text{in}} - 2\,(\hat{\boldsymbol{s}}_{\text{in}} \cdot \hat{\boldsymbol{n}})\,\hat{\boldsymbol{n}}}$$

This formula says "flip the normal component." It encodes the law of reflection (angle in equals angle out) without trigonometry.

Before Figure 5.5, note how the equal angles are measured from the normal.

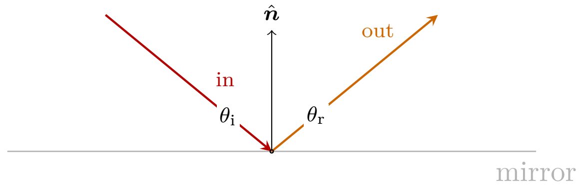


Figure 5.5: Specular reflection: equal angles from the normal; vectors make the rule $\hat{\boldsymbol{s}}_{\text{out}} = \hat{\boldsymbol{s}}_{\text{in}} - 2(\hat{\boldsymbol{s}}_{\text{in}} \cdot \hat{\boldsymbol{n}})\hat{\boldsymbol{n}}$.

Mini Proof

Decompose the incoming direction into parts parallel/perpendicular to the surface: $\hat{\boldsymbol{s}}_{\text{in}} = \hat{\boldsymbol{s}}_{\parallel} + (\hat{\boldsymbol{s}}_{\text{in}} \cdot \hat{\boldsymbol{n}})\,\hat{\boldsymbol{n}}$. Reflection flips only the normal part, so $\hat{\boldsymbol{s}}_{\text{out}} = \hat{\boldsymbol{s}}_{\parallel} - (\hat{\boldsymbol{s}}_{\text{in}} \cdot \hat{\boldsymbol{n}})\,\hat{\boldsymbol{n}}$, which simplifies to the boxed formula above.

5.4 Straight-Line and Curvilinear Motion

For straight motion at constant direction, $\boldsymbol{v}$ is constant in direction and $\boldsymbol{a}$ is parallel (speeding up) or antiparallel (slowing down). Curvilinear motion bends the path: $\boldsymbol{v}$ points along the tangent; $\boldsymbol{a}$ generally has a tangent part (changing speed) and a normal part (turning). At constant speed, acceleration is purely normal and points inward.

As illustrated in Figure 5.6, the normal component indicates how sharply we turn. Quantitatively,

$$\boldsymbol{a} = \underbrace{\dot{v}}_{a_t}\,\hat{\boldsymbol{t}} + \underbrace{v^2\,\kappa}_{a_n}\,\hat{\boldsymbol{n}}, \qquad \kappa = \left\| \frac{\mathrm{d}\hat{\boldsymbol{t}}}{\mathrm{d}s} \right\|.$$

Here $a_t = \dot{v}$ changes speed and $a_n = v^2\kappa$ bends the path. For a circle of radius R, the curvature $\kappa = 1/R$ so $a_n = v^2/R$—the centripetal acceleration used in Chapter 6.

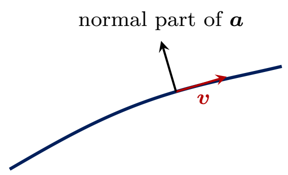


Figure 5.6: At a point on a curve, velocity is tangent; acceleration can have a normal component pointing inward (turning) even at constant speed.

5.5 Examples

Let's connect the pictures to daily motion. Each item hints at $\boldsymbol{r}(t)$, $\boldsymbol{v}(t)$, and $\boldsymbol{a}(t)$ without equations first, encouraging a sketch-then-compute workflow:

- A pedestrian turns a corner at nearly constant speed—$\|\boldsymbol{v}\|$ is steady but $\boldsymbol{a}$ points toward the center of the turn.
- A drone flies north while rising: $x(t)$ and $y(t)$ evolve independently; the path is a tilted line in 3D.
- A robot traces a parametric curve: $\boldsymbol{r}(t) = (t, \sin t)$; the tangent arrows flip as the sine wave crests and troughs.

5.6 Exercises

Practice translating between pictures, components, and geometric statements.

1. **Components.** For $\boldsymbol{r}(t) = (2t, t^2)$, compute $\boldsymbol{v}$ and $\boldsymbol{a}$. At $t = 1$, sketch the tangent.
2. **Dot product.** For $\boldsymbol{a} = (2, 1)$ and $\boldsymbol{b} = (1, 3)$, compute $\boldsymbol{a} \cdot \boldsymbol{b}$ and the angle between them.
3. **Practical: Corner turn.** Walk at steady speed and turn a corner; note the inward "pull". Sketch $\boldsymbol{v}$ and the inward normal direction.
4. **Parametric reading.** For $\boldsymbol{r}(t) = (t, \sin t)$, mark where $\boldsymbol{v}$ is horizontal or vertical.
5. **Area/work preview.** A constant force $\boldsymbol{F} = (1, 0)$ moves a point along $\boldsymbol{r}(t) = (t, t^2)$. Compute $\int_0^1 \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r}$.

5.7 Summary and Review

A quick checklist before moving on:

- Vector kinematics: $\boldsymbol{r}$, $\boldsymbol{v} = \dot{\boldsymbol{r}}$, $\boldsymbol{a} = \dot{\boldsymbol{v}}$; components evolve independently.
- Tangent/normal picture: $\boldsymbol{v}$ tangent; $\boldsymbol{a}$ can turn you even at constant speed.
- Dot product measures alignment and powers work; projections simplify problems.

5.8 Where We're Heading Next

In Chapter 6 we apply vector kinematics to parabolic trajectories without air and to uniform/non-uniform circular motion with centripetal acceleration and geometric decompositions. The tangent/normal picture from Figure 5.6 will become your compass.

Common Pitfalls

Quick reminders:

- Adding vector magnitudes instead of components; project first.
- Forgetting that $\boldsymbol{v}$ is tangent and $\boldsymbol{a}$ can be normal even at constant speed.
- Mixing degrees and radians in parametric angles.

Try in 60 seconds

Quick practice:

- Draw a random curve and sketch $\hat{\boldsymbol{t}}$ and $\hat{\boldsymbol{n}}$ at one point.
- Add two 2D arrows and estimate the angle between them by eye, then compute via dot product.
- Project a vector onto the x-axis and check that lengths match.

Chapter 6

Projectiles and Circular Motion

This chapter applies vector kinematics to two iconic trajectories: parabolas (projectiles without air) and circles (uniform and non-uniform circular motion). We connect algebraic formulas to clean geometric pictures and keep the tangent/normal viewpoint from Chapter 5 front and center. Our strategy is Galileo's: split motion into independent directions, solve the easy parts, and then recombine.

Learning Objectives

You will derive and interpret projectile trajectories, read uniform circular motion parameters, and decompose acceleration into tangent and normal parts for non-uniform turns.

Symbols at a Glance

v_0 launch speed, α launch angle, g gravitational acceleration, R radius, ω angular speed, $\hat{\boldsymbol{t}}$ tangent unit, $\hat{\boldsymbol{n}}$ inward normal unit.

Analogy: Two Classics

Projectiles are the "throw and arc" stories; circular motion is the "turn and whirl" story. In both, the red velocity arrows kiss the path (tangent), and inward acceleration keeps the shape honest.

6.1 Projectile Motion Without Air

Resolve the initial velocity $\boldsymbol{v}_0$ into components: $v_{0x} = v_0 \cos\alpha$, $v_{0y} = v_0 \sin\alpha$. With x horizontal and y vertical, the equations are

$$x(t) = x_0 + v_{0x}\, t,$$
$$y(t) = y_0 + v_{0y}\, t - \tfrac{1}{2} g t^2, \qquad g > 0.$$

Eliminating t gives a parabola. The time of flight (for $y_0 = 0$) is $T = \frac{2v_0 \sin\alpha}{g}$; the range is $R = \frac{v_0^2 \sin 2\alpha}{g}$. Two independent "clocks" are running: a steady horizontal clock $x(t) = x_0 + v_{0x}t$ and a vertical clock slowed by gravity $y(t) = y_0 + v_{0y}t - \frac{1}{2}gt^2$.

For level launch and landing ($y_0 = 0$), the range is maximized at $\alpha = 45°$ (no air). If launch and landing heights differ, the optimal angle shifts; independence of horizontal and vertical motion holds only when air resistance is neglected and gravity is uniform.

Independent Clocks

Think of the projectile as walking east with a steady pace (horizontal component) while riding an elevator up and then down (vertical component). The two stories are independent, and the combined motion is the diagonal arc you see.

As illustrated in Figure 6.1, different launch angles trace a family of parabolas; complementary angles share the same range.

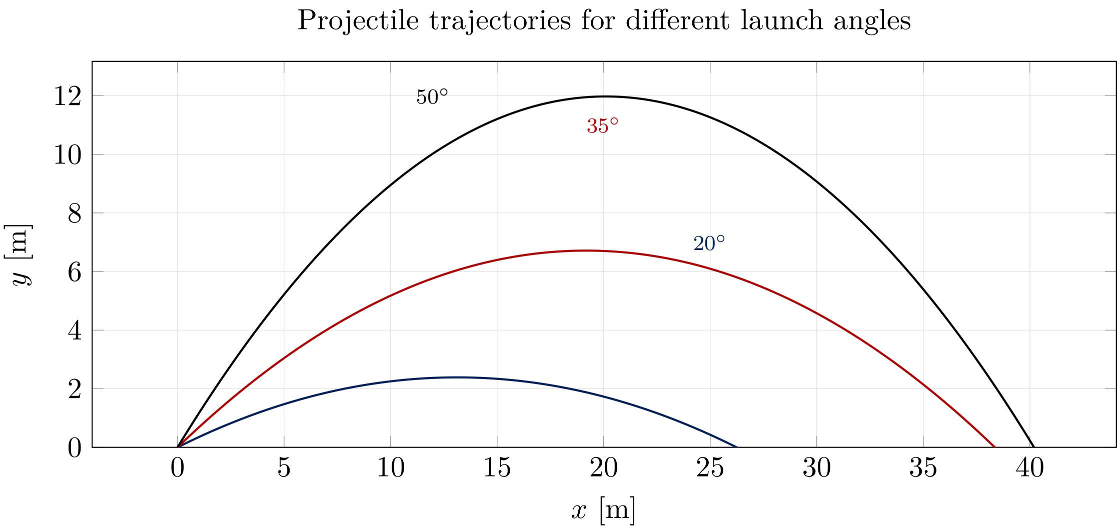


Figure 6.1: Projectile trajectories for three launch angles with the same initial speed v_0. Complementary angles (e.g., 35° and 55°) have equal range.

Worked Example: Free Kick

Kick a ball at $v_0 = 25\,\mathrm{m/s}$ and $\alpha = 30°$ from ground level. Neglecting air, the flight time is $T = \frac{2v_0 \sin\alpha}{g} \approx 2\,(25 \sin 30°)/9.81 \approx 2.55\,\mathrm{s}$. The range is $R = \frac{v_0^2 \sin 2\alpha}{g} \approx (25^2 \sin 60°)/9.81 \approx 55.3\,\mathrm{m}$. The apex height $h_{\mathrm{max}} = \frac{v_0^2 \sin^2\alpha}{2g} \approx 8.0\,\mathrm{m}$. Figure 6.2 shows the arc with markers at the apex and landing.

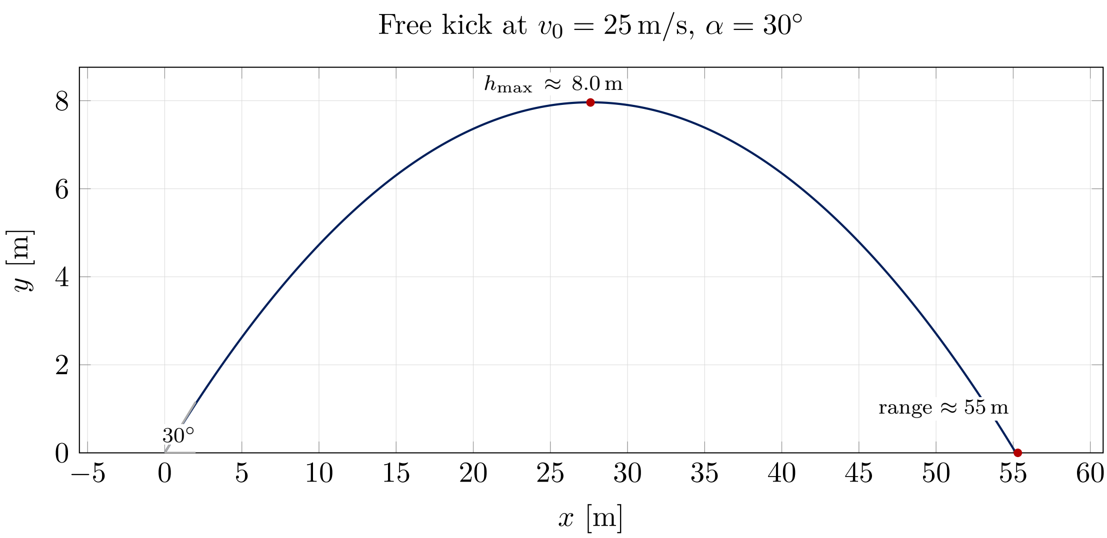


Figure 6.2: Projectile path for the free-kick parameters. Dots mark the apex and landing point.

6.2 Uniform Circular Motion

Parametrize a circle of radius R by $\boldsymbol{r}(t) = \big(R\cos(\omega t),\, R\sin(\omega t)\big)$. Then

$$\boldsymbol{v}(t) = \dot{\boldsymbol{r}}(t) = \omega R\big(-\sin\omega t, \cos\omega t\big), \qquad \boldsymbol{a}(t) = \dot{\boldsymbol{v}}(t) = -\omega^2 R\big(\cos\omega t, \sin\omega t\big) = -\omega^2 \boldsymbol{r}(t).$$

The acceleration points inward (centripetal) with magnitude $a = \omega^2 R = v^2/R$. Use radians for ω and any calculus with angles. At constant speed, the compass (direction) turns even while the speedometer reading stays fixed—acceleration measures turning, not only speeding up.

Before Figure 6.3, note that velocity is tangent and acceleration is inward at every point. A tight rope swing or a car on a roundabout are everyday versions of the same geometry: something must pull inward to bend the path.

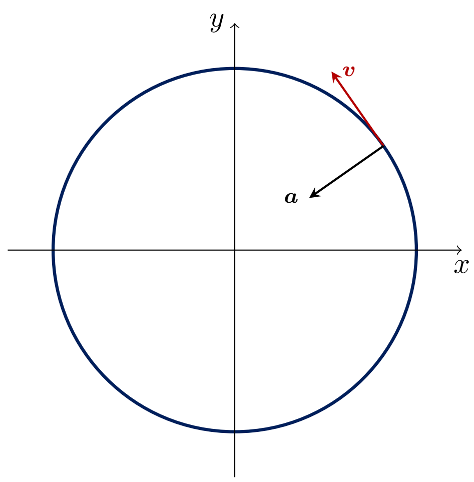


Figure 6.3: Uniform circular motion: velocity is tangent; acceleration points inward (centripetal).

Worked Example: Ferris Wheel

A seat on a Ferris wheel of radius $R = 12\,\text{m}$ completes one revolution in $T = 8.0\,\text{s}$. Then $\omega = 2\pi/T \approx 0.785\,\text{rad/s}$, speed $v = \omega R \approx 9.42\,\text{m/s}$, and centripetal acceleration $a_n = v^2/R \approx 7.4\,\text{m/s}^2$. At the top and bottom the acceleration is vertical (downward at top, upward at bottom). Figure 6.4 shows $\boldsymbol{v}$ (tangent) and $\boldsymbol{a}$ (inward) at both points.

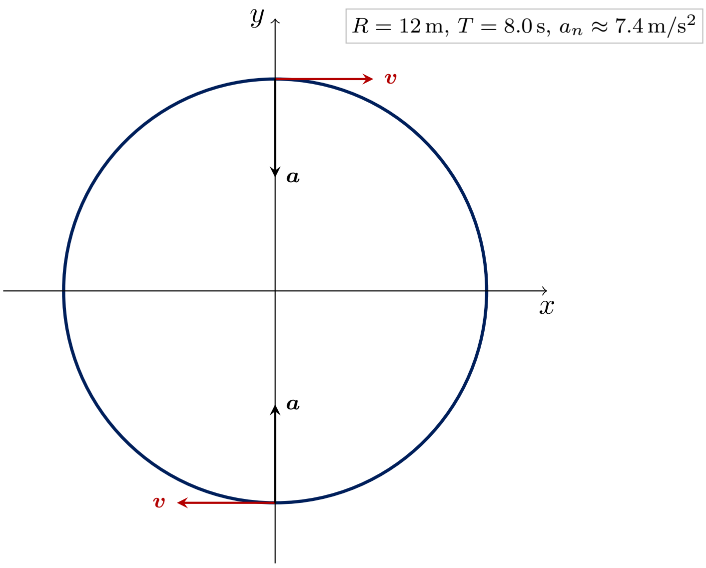


Figure 6.4: Ferris wheel kinematics: tangent velocity and inward (centripetal) acceleration at the top and bottom.

6.3 Non-Uniform Circular Motion

When speed changes along a curve, acceleration splits into tangent and normal parts,

$$\boldsymbol{a} = a_{\text{t}}\,\hat{\boldsymbol{t}} + \frac{v^2}{R}\,\hat{\boldsymbol{n}}, \qquad a_{\text{t}} = \dot{v},$$

where $\hat{\boldsymbol{t}}$ is the unit tangent and $\hat{\boldsymbol{n}}$ points inward. As previewed in Figure 6.5, pressing the accelerator while turning adds a forward (tangent) component to the always-inward normal part.

Local Runway

At any point the road looks straight for a tiny patch. We set up a *local runway* with a forward axis $\hat{\boldsymbol{t}}$ and an inward axis $\hat{\boldsymbol{n}}$. The velocity lies along the runway; the acceleration splits into a forward part (change of speed) and an inward part (change of direction).

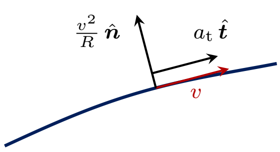


Figure 6.5: Decomposition of acceleration into tangent and normal components at a point on a curve: velocity v is tangent; $a_{\text{t}}\,\hat{\boldsymbol{t}}$ changes speed; $(v^2/R)\,\hat{\boldsymbol{n}}$ bends the path inward.

6.4 Exercises

Practice reading vector motion in the plane and circle.

1. **Parabola checks.** For $v_0 = 20\,\mathrm{m/s}$ and $\alpha = 30°$, compute T, R, and the maximum height; compare with the curve in Figure 6.1.
2. **Complementary angles.** Show that $\sin 2\alpha$ is unchanged by $\alpha \mapsto 90° - \alpha$; interpret the equal-range property.
3. **Uniform circle.** For $R = 2\,\mathrm{m}$ and $\omega = 1.5\,\mathrm{rad/s}$, compute v and a; mark the vectors on a sketch like Figure 6.3.
4. **Practical: Hose arc.** Observe a water stream from a garden hose (low speed, calm day). Sketch the curve and mark where the vertical component of velocity changes sign.
5. **Practical: Cornering.** On a bike at steady speed, ride a gentle arc; identify the inward "pull" as the normal component of $\boldsymbol{a}$.

6.5 Summary and Review

A quick checklist before moving on:

- Projectiles: split into components, solve, and recombine; parabolas and range emerge cleanly.
- Uniform circles: $\boldsymbol{v}$ is tangent; $\boldsymbol{a}$ is inward with magnitude v^2/R.
- Non-uniform turning: decompose acceleration into tangent and normal parts.

6.6 Where We're Heading Next

In Chapter 7 we blend vector kinematics with force models to analyze motion in two and three dimensions—free-body diagrams in 2D, constraints like tension and normal forces, and energy methods as an alternative lens.

Common Pitfalls

Quick cautions:

- Treating the speed v as zero at the projectile apex (only $v_y = 0$ there).
- Ignoring component independence—horizontal and vertical evolve separately without coupling forces.
- Mixing degrees with radians in trigonometric functions.

Try in 60 seconds

Speed drills:

- Write $x(t), y(t)$ for a projectile with v_0 and α.
- On a circle, draw $\boldsymbol{v}$ and inward $\boldsymbol{a}$ at one point.
- Split an arbitrary $\boldsymbol{a}$ into tangent and normal directions on a crude sketch.

Part IV

Forces, Work, and Energy

Part IV Overview

Forces meet vectors in multiple dimensions. Chapter 7 develops free-body diagrams in 2D, resolves vector laws into components, and models friction, drag, and tension. Chapter 8 then introduces work, kinetic energy, and power as complementary tools that often simplify multi-step force problems. Chapter 9 completes the part with potential energy and energy conservation, using energy diagrams to read turning points and speeds at a glance.

Chapter 7

Forces in Multiple Dimensions

We now combine Newton's laws with vectors. In two dimensions the recipe is clear: isolate the object, draw a free-body diagram (FBD), write the vector balance $\sum \boldsymbol{F} = m\,\boldsymbol{a}$, and then resolve into axes that make the geometry simple.

Learning Objectives

By the end, you will draw clean 2D FBDs, choose helpful axes, resolve forces into components, and write the pair of equations that come from $\sum F_x = ma_x$ and $\sum F_y = ma_y$.

Symbols at a Glance

$\boldsymbol{F}$ force, $\boldsymbol{a}$ acceleration, m mass, $\hat{\boldsymbol{t}}$ axis tangent to a surface, $\hat{\boldsymbol{n}}$ axis normal to a surface, N normal force, $W = mg$ weight, f friction, T tension.

Analogy: Shadows and Balances

Imagine each force arrow casting a shadow on your chosen axes. The component equations simply say: the right-minus-left shadow balances to ma_x, and the up-minus-down shadow balances to ma_y. Choose axes so the important shadows fall cleanly.

7.1 Newton's Laws in Vector Form

We keep the vector law in sight and then read its "shadows" on the axes we choose. The second law reads

$$\sum \boldsymbol{F} = m\,\boldsymbol{a} \quad \text{(vectors)}.$$

Pick axes and project:

$$\sum F_x = ma_x, \qquad \sum F_y = ma_y.$$

The art is choosing axes aligned with the geometry—on an incline, take one axis along the surface (Figure 7.1). With a good choice, one component often vanishes (e.g., no acceleration into a rigid surface).

7.2 Free-Body Diagrams in 2D

An FBD strips the world down to just the object and the forces on it. Draw arrows from the object's center pointing in the directions of the forces, label magnitudes, and note your axis choice. A neat diagram saves algebra. Figure 7.1 shows a block on an incline with the usual suspects.

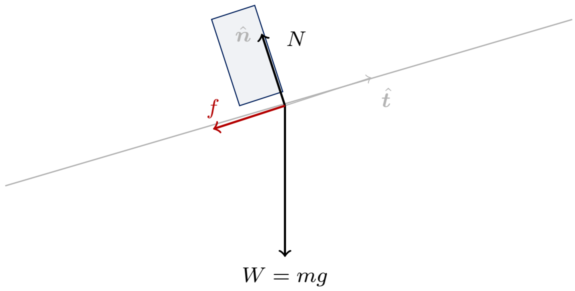


Figure 7.1: Free-body diagram on an incline: axes chosen along tangent $\hat{\boldsymbol{t}}$ and normal $\hat{\boldsymbol{n}}$; forces are weight W, normal N, and friction f (kinetic shown opposing motion).

7.3 Resolving Components

Choose axes along the surface $\hat{\boldsymbol{t}}$ (take the downhill direction as positive) and perpendicular to it $\hat{\boldsymbol{n}}$ (pointing outward from the surface). In these axes, the weight W splits cleanly: a component of magnitude $W \sin\theta$ lies along $+\hat{\boldsymbol{t}}$ (downhill) and a component of magnitude $W \cos\theta$ points into the surface (opposite $+\hat{\boldsymbol{n}}$). The angle θ is the incline angle. The faint gray arrows in Figure 7.2 indicate the chosen axes.

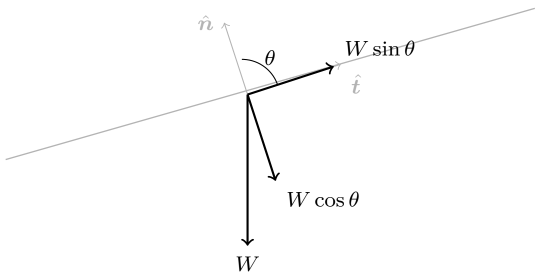


Figure 7.2: Resolving weight on an incline with axes shown faint gray: $W \sin\theta$ along $+\hat{\boldsymbol{t}}$ (downhill) and $W \cos\theta$ into the surface (opposite $+\hat{\boldsymbol{n}}$).

7.4 Friction and Drag

Static friction adjusts to whatever is needed to prevent relative motion *up to a limit*:

$$|f_{\mathrm{s}}| \leq \mu_{\mathrm{s}} N, \quad \text{with } f_{\mathrm{s}} = \mu_{\mathrm{s}} N \text{ at incipient slip.}$$

Once sliding, kinetic friction has nearly fixed magnitude $f_{\mathrm{k}} = \mu_{\mathrm{k}} N$ opposing motion. Linear drag models air/fluid resistance at low speeds as $\boldsymbol{F}_{\mathrm{drag}} = -c\,\boldsymbol{v}$; direction opposes $\boldsymbol{v}$ and c lumps density/shape factors (see Chapters 14 and 15). Start every problem by deciding: static (sticking), kinetic (sliding), or a low-speed drag model.

Analogy: Grip vs. Slide

Static friction is a grippy shoe matching whatever gentle push you give up to a limit; kinetic friction is the steady resistance once you're sliding.

Constraint forces round-out the toolkit: the normal force enforces no-penetration, and ideal tension acts along a string/rope. In the common idealization of massless strings and frictionless pulleys, the tension has the same magnitude throughout a connected segment.

7.5 From Vector Law to Equations

Goal: turn a picture into two clean equations you can solve. The approach is always the same: decide on axes, write the vector law, project onto axes, and use any constraint equations (like the no–lift-off condition) to eliminate unknown forces.

A reliable four-step recipe (use it every time):

1. Draw a neat FBD and *choose axes* that match the geometry (e.g., along and normal to a surface).
2. Write the *vector* law $\sum \boldsymbol{F} = m\,\boldsymbol{a}$ with a sentence about the positive directions.
3. *Project* onto axes to get $\sum F_t = ma_t$ and $\sum F_n = ma_n$ (or x/y in a standard frame).
4. Use *constraints* (e.g., $a_n = 0$ with no lift-off, $f = \mu_k N$ when sliding) to eliminate unknowns, then solve for the desired quantity (a_t, $v(t)$, or $x(t)$).

For the incline with kinetic friction ($f = \mu_k N$) and angle θ measured from horizontal, choosing $\hat{\boldsymbol{t}}$ downhill and $\hat{\boldsymbol{n}}$ outward gives

$$\begin{aligned} \hat{\boldsymbol{t}}:&\quad ma_t = W\sin\theta - f,\\ \hat{\boldsymbol{n}}:&\quad 0 = N - W\cos\theta \quad \text{(no lift-off)}. \end{aligned}$$

Eliminate N using the normal equation; then $a_t = g\sin\theta - \mu_k g\cos\theta$. From here, constant a_t lets you reuse the kinematics of Chapter 3 for $v(t)$ and $x(t)$ along the slope.

Worked Example: Sliding Down an Incline

Start from the free-body picture along the slope. With axes $\hat{\boldsymbol{t}}$ (down the surface) and $\hat{\boldsymbol{n}}$ (out of the surface), Newton's second law along $\hat{\boldsymbol{t}}$ reads

$$ma_t = mg\sin\theta - f, \qquad f = \mu_k N, \qquad N = mg\cos\theta,$$

so the constant downhill acceleration is

$$\boxed{a_t = g\big(\sin\theta - \mu_k\cos\theta\big).}$$

The motion threshold occurs when $a_t = 0$, i.e. $\tan\theta_c = \mu_k$. For $\mu_k = 0.25$, $\theta_c \approx 14.0^\circ$ and $a_t(20^\circ) \approx 1.05\,\mathrm{m/s^2}$. From rest over $s = 5.0\,\mathrm{m}$, $v = \sqrt{2a_t s} \approx 3.24\,\mathrm{m/s}$. Figure 7.3 shows the FBD; the caption states $a_t(\theta)$ and the threshold condition.

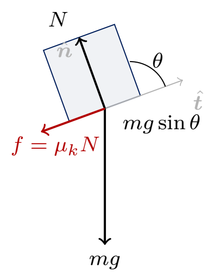


Figure 7.3: Free-body diagram on an incline with axes along/normal to the surface; friction opposes motion. The downhill acceleration is $a_t(\theta) = g(\sin\theta - \mu_k\cos\theta)$; motion starts when $a_t > 0$, i.e. $\theta > \theta_c = \tan^{-1}\mu_k$ (for $\mu_k = 0.25$, $\theta_c \approx 14^\circ$).

7.6 Exercises

Quick practice with FBDs and components.

1. **Axes choice.** For a block on an incline, draw $\hat{\boldsymbol{t}}$, $\hat{\boldsymbol{n}}$, and all forces. Label signs.
2. **Resolve weight.** Show that the downhill component is $W \sin\theta$ and the into-surface component is $W \cos\theta$.
3. **Static or kinetic?** A block rests on an incline and you increase the angle slowly. Describe what friction does before and after it starts sliding.
4. **Practical: Door push.** Push a door near the hinge and then near the handle with the same effort. Describe the difference in motion; sketch the force directions (torque preview).
5. **Drag direction.** For a puck moving east over a thin oil film, draw $\boldsymbol{v}$ and $\boldsymbol{F}_{\text{drag}}$.

7.7 Summary and Review

A quick checklist before moving on:

- Vector law $\sum \boldsymbol{F} = m\,\boldsymbol{a}$ becomes two component equations once axes are chosen.
- FBDs isolate forces; axes aligned with geometry simplify algebra.
- Friction and drag oppose motion; normal forces constrain; tension pulls along strings.

7.8 Where We're Heading Next

In Chapter 8 we'll solve problems by tracking energy transfer—often shorter and more insightful than summing forces step by step.

Common Pitfalls

Quick cautions:

- Mixing axis components when projecting $\sum \boldsymbol{F} = m\,\boldsymbol{a}$; treat t/n (or x/y) separately.
- Assuming friction always equals μN—static friction adjusts up to a limit.
- Forgetting that normal forces enforce constraints and may vary with situation.

Try in 60 seconds

Tiny drills:

- Draw an FBD for a hanging mass with a side pull; mark components.
- Write $\sum F_x = ma_x$, $\sum F_y = ma_y$ for the incline case and solve for a_t.
- Point at any moving object around you and say which forces act and in what directions.

Chapter 8

Work, Kinetic Energy, and Power

Energy is a second viewpoint on motion that often turns multi-step force calculations into one line. When forces are tricky but you can track start and finish, energy methods shine.

Learning Objectives

You will compute work from force and displacement (including as an area), use the work–kinetic energy theorem to relate forces to speed, and interpret power as the rate at which work is done.

Symbols at a Glance

W work (joules), $K = \frac{1}{2}mv^2$ kinetic energy, P power, $\boldsymbol{F}$ force, $\mathrm{d}\boldsymbol{r}$ displacement element, F_t component of $\boldsymbol{F}$ along the displacement.

Analogy: Battery and Faucet

Think of kinetic energy as a battery level. Doing *positive work* charges it; *negative work* drains it. Power is the faucet setting—how fast you add or remove "charge." A gentle trickle (low power) eventually fills the battery; a wide-open faucet (high power) fills it quickly.

8.1 Work as Force Along a Displacement

For a displacement from $\boldsymbol{r}_1$ to $\boldsymbol{r}_2$ under force $\boldsymbol{F}(\boldsymbol{r})$, the work done *by the force* is

$$W = \int_{\boldsymbol{r}_1}^{\boldsymbol{r}_2} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} = \int_{s_1}^{s_2} F_t(s)\,\mathrm{d}s,$$

where F_t is the tangential component along the path coordinate s. Geometrically, the dot captures "how much the force points along the motion." For a constant tangential force over a straight path of length Δs, this reduces to $W = F_t\,\Delta s$.

Sign and direction matter: only the component of force *along* the displacement contributes. Forces perpendicular to the motion (e.g., an ideal normal force that enforces contact without slip) do *no* work because $\boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} = 0$.

As illustrated in Figure 8.1, work is the *area* under the force–displacement curve F_t vs. s between the start and end points. When $\boldsymbol{F}$ is conservative, this area depends only on the endpoints and defines a potential energy difference ΔU (Chapter 9).

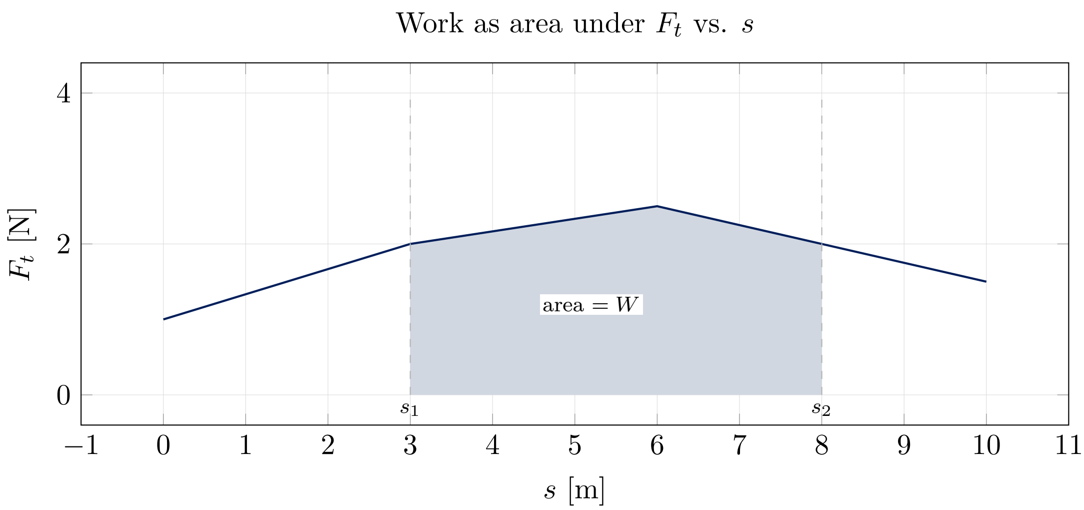


Figure 8.1: Work equals the shaded area under the tangential force F_t versus displacement s from s_1 to s_2.

Worked Example: Stair Climb

A 70 kg person climbs one floor of height $h = 3.0\,\text{m}$ in 6.0 s. The work against gravity is $W \approx mgh = 70 \times 9.81 \times 3.0 \approx 2.06\,\text{kJ}$. The average power is $\bar{P} = W/\Delta t \approx 2.06\,\text{kJ}/6.0\,\text{s} \approx 343\,\text{W}$. Figure 8.2 summarizes the numbers.

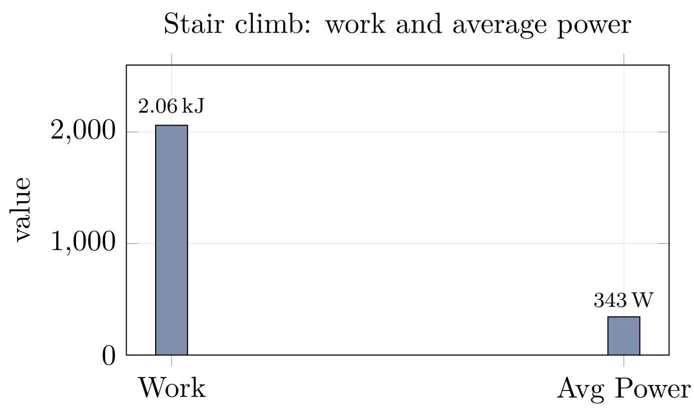


Figure 8.2: Work and average power for a one-floor stair climb.

8.2 Work–Kinetic Energy Theorem

The net work done on a particle changes its kinetic energy:

$$W_{\text{net}} = \Delta K = K_2 - K_1 = \tfrac{1}{2}mv_2^2 - \tfrac{1}{2}mv_1^2.$$

Sketch-first reasoning: if the net force does positive work, the speed increases; negative work reduces speed. This turns force stories into speed stories without tracking every instant.

This statement holds regardless of whether forces are conservative or not; "net work" collects contributions from all forces. When forces are conservative, we can package their work into a potential U and trade W_{net} for changes in $K + U$ (Chapter 9).

Analogy: Energy Ledger

Think of kinetic energy as a bank balance of motion. Positive work is a deposit; negative work is a withdrawal. The statement "work equals change in kinetic energy" is just the ledger balancing.

To visualize the nonlinearity of kinetic energy with speed, Figure 8.3 plots $K(v)$ and marks two speeds.

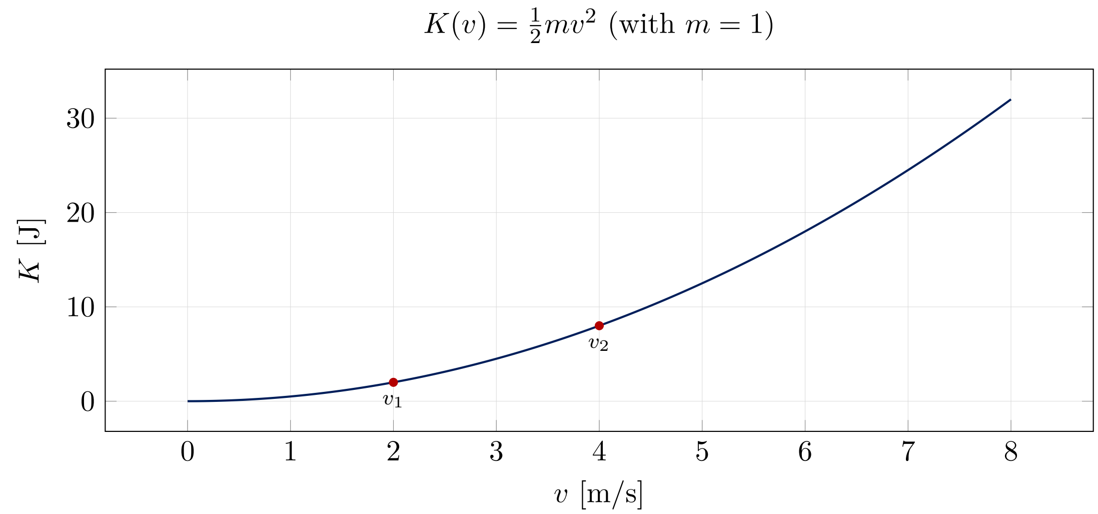


Figure 8.3: Kinetic energy grows quadratically with speed: doubling speed quadruples K.

8.3 Power

Power is the *instantaneous* rate at which work is done: $P = \frac{\mathrm{d}W}{\mathrm{d}t} = \boldsymbol{F} \cdot \boldsymbol{v}$. A high power delivery changes kinetic energy quickly; a low power delivery takes time. Constant power means equal areas in $P(t)$ give equal chunks of work. In rotational problems (Chapter 11), the analogous relation is $P = \tau\,\omega$.

As a simple illustration, Figure 8.4 shows a short interval of nearly constant power and the work accumulated as the area under $P(t)$.

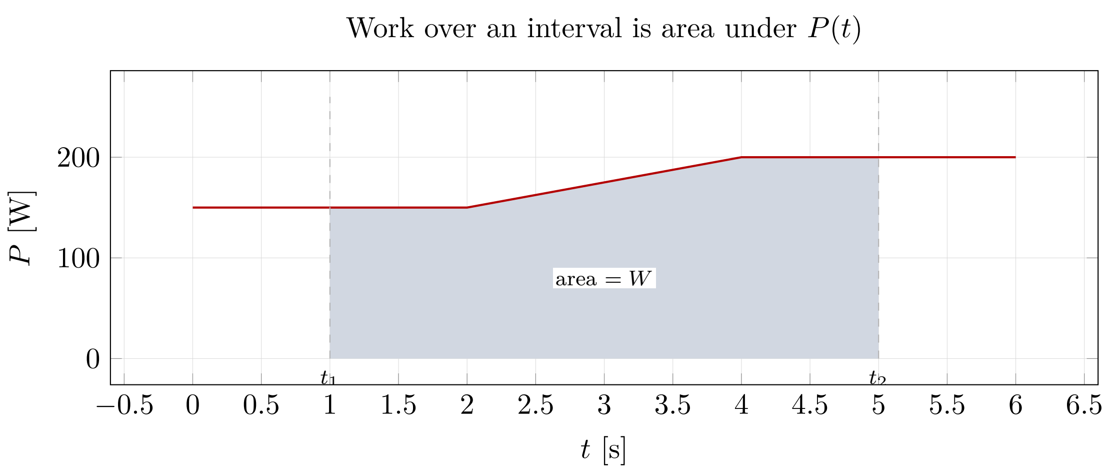


Figure 8.4: Power over time: the shaded area under $P(t)$ equals the work done between t_1 and t_2.

8.4 Worked Example: Same Work Two Ways

Consider a push of constant tangential force $F_t = 100\,\mathrm{N}$ over a straight $\Delta s = 3\,\mathrm{m}$. The work from the force–distance view is

$$W = F_t\,\Delta s = 100 \times 3 = 300\,\mathrm{J}.$$

If the motion happens at a steady speed $v = 1.5\,\mathrm{m/s}$ for $\Delta t = 2\,\mathrm{s}$, the power is constant $P = F_t v = 150\,\mathrm{W}$ and the work from the power–time view is

$$W = \int_{t_1}^{t_2} P\,\mathrm{d}t = P\,\Delta t = 150 \times 2 = 300\,\mathrm{J}.$$

Both paths agree, as shown in Figure 8.5.

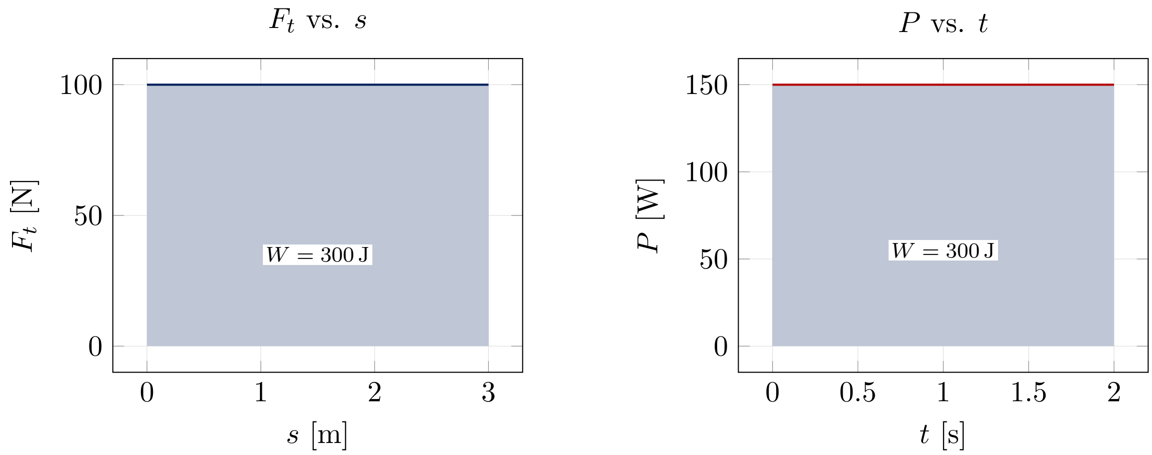


Figure 8.5: The same work computed two ways: area $F_t\,\Delta s$ under force–distance, and area $P\,\Delta t$ under power–time.

8.5 Everyday Energy Examples

Energy reasoning excels when only the start and finish matter. Three favorites:

- **Stair climb:** Work $\approx mgh$; power is higher when you sprint.
- **Bike up a hill:** Same vertical gain h gives the same mgh of work, regardless of path; timing changes power.
- **Car speeding up:** Engine power raises K; doubling speed takes four times the energy.

8.6 Exercises

Quick practice connecting forces, work, energy, and power.

1. **Units.** Check the SI units of work, energy, and power.
2. **Area and work.** For the piecewise force in Figure 8.1, estimate the shaded area (work) by counting grid squares.
3. **Kinetic energy growth.** Using Figure 8.3, explain why doubling speed takes four times the energy.
4. **Practical: Stair climb.** Estimate the work to climb one floor; compare power for an easy pace vs. a sprint.
5. **Power area.** On Figure 8.4, compute the shaded work between t_1 and t_2.

8.7 Summary and Review

A quick checklist before moving on:

- Work is the integral of tangential force along a displacement (area under F_t vs. s).
- Net work changes kinetic energy: $\Delta K = W_{\text{net}}$.
- Power is the rate of doing work: areas under $P(t)$ accumulate work ($P = \tau\,\omega$ in rotation; Chapter 11).
- Perpendicular forces do no work; the sign of work follows alignment with motion.

Forces or Energy?

Choosing a method, made simple:

- Use **energy** for speeds at positions or turning points, when only the start and finish matter; path-independent when forces are conservative.
- Use **forces** (and components) for directions, time evolution, or when non-conservative forces dominate.
- Mix them wisely: forces set up constraints and signs; energy closes the speed/height loop.

8.8 Where We're Heading Next

In Chapter 9 we combine work and kinetic energy with potential energy and the conservation law $K + U = \text{const}$ for conservative forces; then we use energy diagrams to read turning points.

Common Pitfalls

Avoid these slips:

- Forgetting that only *net* work changes kinetic energy.
- Using energy conservation when non-conservative work is significant without accounting for it.
- Dropping the dot product: only the component of force along motion contributes to work.

Try in 60 seconds

Tiny wins using the new tools:

- Point at a moving object and say whether the net work on it is likely positive, negative, or near zero over the next second—and why.
- Sketch any F_t vs. s piecewise-constant curve and shade the work between two marks.
- Double a speed in your head and state what happens to K.
- Write $P = \boldsymbol{F} \cdot \boldsymbol{v}$ for a car pushing forward with force F at speed v.

Chapter 9

Potential Energy and Conservation of Energy

For certain forces, we can store their "influence" in a scalar function U called potential energy. When only such conservative forces act, the total mechanical energy $E = K + U$ stays constant and the motion traces out the landscape of U.

Learning Objectives

You will identify conservative forces, compute and sketch common potentials, use $E = K+U$ to find speeds and turning points, and read energy diagrams for qualitative motion.

Symbols at a Glance

U potential energy, $K = \frac{1}{2}mv^2$ kinetic energy, E total mechanical energy, $\boldsymbol{F}$ force, $F_x = -\,\mathrm{d}U/\mathrm{d}x$ in 1D, k spring constant, g gravitational acceleration.

Analogy: Height Map

Think of U as a height map. Mechanical energy $E = K + U$ is like hiking with a fixed budget: when you go "higher" on the map (larger U), your running energy K must drop; when the trail goes down, K increases. Turning points occur where your budget just equals the height.

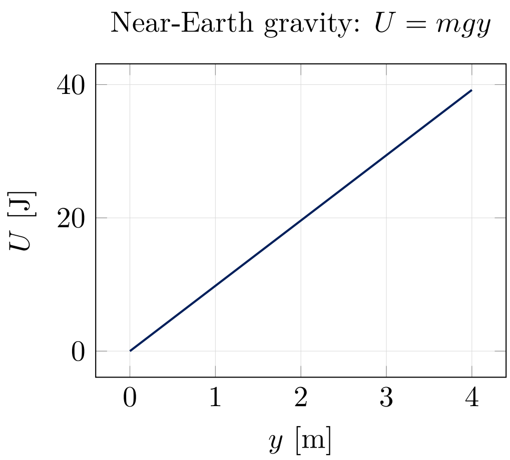
9.1 Conservative Forces and Potentials

In one dimension, a force is conservative if it can be written as $F_x(x) = -\,\mathrm{d}U/\mathrm{d}x$ for some $U(x)$. In multiple dimensions, conservative means $\boldsymbol{F} = -\nabla U$; equivalently, the work around any closed loop is zero. In smooth, simply connected regions this is the same as $\nabla \times \boldsymbol{F} = \mathbf{0}$ (see Appendix C). Only *differences* in U matter; adding a constant does not change physics.

Two workhorses:

- Near-Earth gravity: $F_y = -mg$ gives $U(y) = mgy$ up to an arbitrary constant.
- Hooke spring: $F_x = -kx$ gives $U(x) = \frac{1}{2}kx^2$.

As previewed in Figure 9.1, one is linear, the other parabolic.



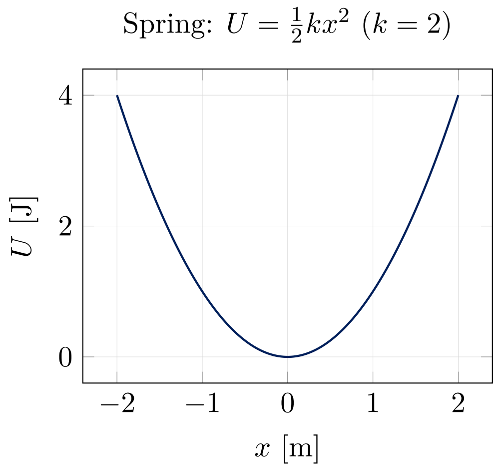


Figure 9.1: Two basic potentials: linear for near-Earth gravity and parabolic for a spring. For inverse-square gravity $U = -GMm/r$, see Chapter 12.

9.2 Energy Conservation and Turning Points

For conservative forces, $E = K + U$ stays constant. On an energy diagram $U(x)$, draw a horizontal line at E; the motion is allowed where $E \geq U(x)$, with speed $v(x) = \sqrt{\frac{2}{m}(E - U(x))}$. Points where $E = U$ are *turning points* ($v = 0$).

The slope of U controls the direction of acceleration: $a(x) = \frac{F_x}{m} = -\frac{1}{m}\frac{\mathrm{d}U}{\mathrm{d}x}$. Where U decreases with x (downhill), $a > 0$; where it increases, $a < 0$. Near a stable equilibrium x_0 (a local minimum of U), a Taylor expansion $U(x) \approx U(x_0) + \frac{1}{2}k(x - x_0)^2$ yields simple harmonic motion with $k = U''(x_0)$ and $\omega = \sqrt{k/m}$ (see Chapter 13).

Analogy: Hiking an Energy Landscape

Imagine hiking on a landscape $U(x)$. Your “height” U plus your “running energy” K must equal the fixed total E. When the trail rises to meet E, you have to stop and turn.

As shown in Figure 9.2, a spring potential with total energy E yields symmetric turning points and the shaded gap between E and U visually represents K. A gravitational version for a vertical toss appears in Figure 9.3.

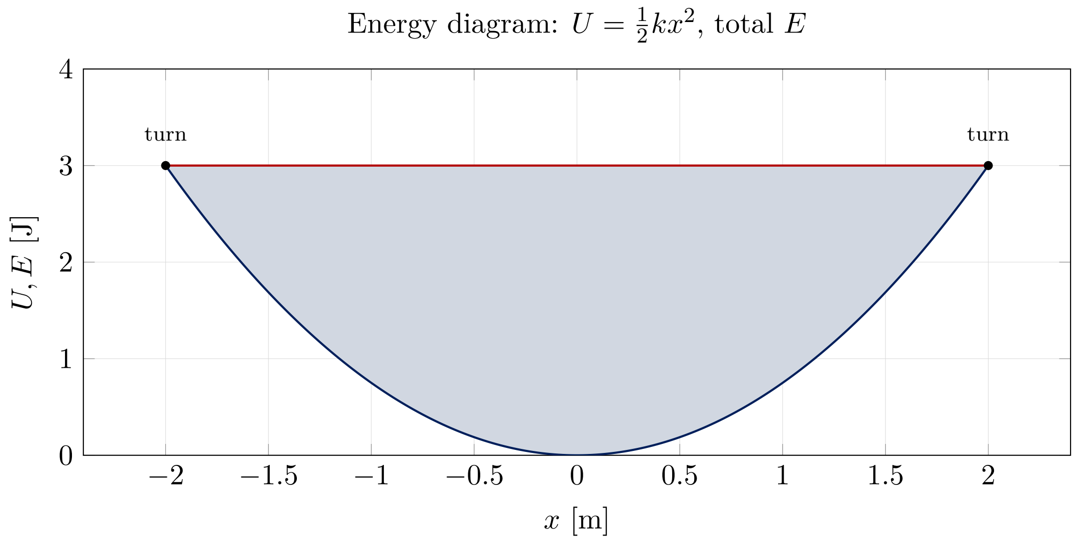


Figure 9.2: Energy diagram for a spring: shaded region represents kinetic energy $K = E - U(x)$; dots mark turning points where $K = 0$.

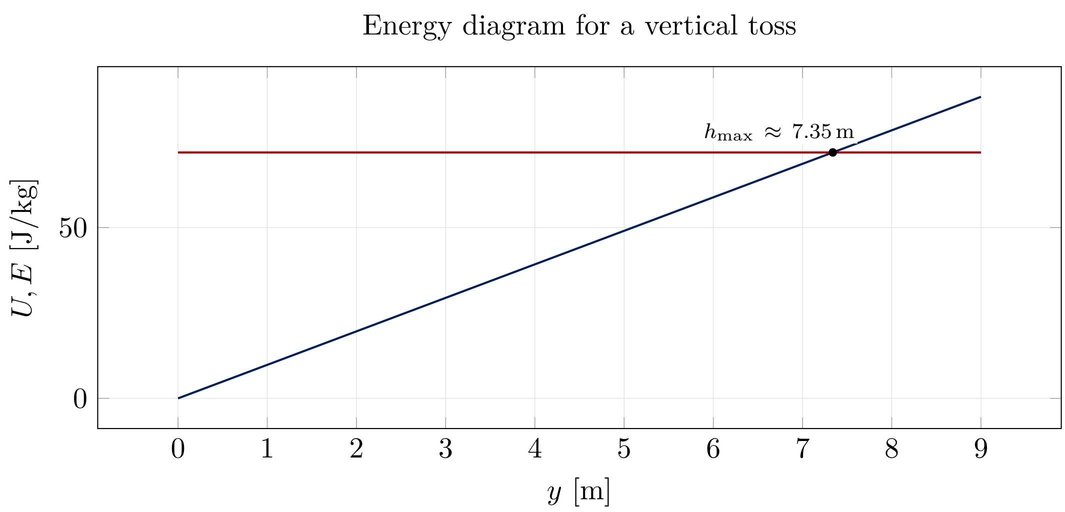


Figure 9.3: Linear gravitational potential per unit mass $U/m = gy$ (navy) and constant total energy per unit mass $E/m = \frac{1}{2}v_0^2$ (red). The intersection marks h_{max}.

Worked Example: Ball Toss via Energy

Throw a ball upward from $y_0 = 0$ with speed $v_0 = 12\,\text{m/s}$ (ignore air). With $U(y) = mgy$ and $E = K_0 + U_0 = \frac{1}{2}mv_0^2$, the apex height h_{max} occurs where $K = 0$, so $E = U$: $mgh_{max} = \frac{1}{2}mv_0^2$ giving $h_{max} = v_0^2/(2g) \approx 7.35\,\text{m}$. The speed at any height follows $v(y) = \sqrt{v_0^2 - 2gy}$. Figure 9.3 shows $U(y)$ and the energy line with h_{max} marked.

9.3 Non-Conservative Forces (Brief)

When friction or drag act, mechanical energy changes according to the work by non-conservative forces: $\Delta E = W_{nc}$. Speeds are then found from $K_2 = K_1 + W_{nc} - \Delta U$. In practice: start with $E_1 = K_1 + U_1$, add the signed work by non-conservative forces to get E_2, and solve $K_2 = E_2 - U_2$ for the speed.

9.4 Examples

Before computing, read the energy picture and predict: where can it move (allowed region), where will it turn, and where is it fastest (largest K)?

- **Mass–spring:** With total energy E, the speed at x follows $v(x) = \sqrt{\frac{2}{m}(E - \frac{1}{2}kx^2)}$.
- **Ball toss (no air):** $U = mgy$, so $v(y) = \sqrt{v_0^2 - 2g(y - y_0)}$; the peak where $v = 0$ solves $E = U$.
- **Slide with friction:** Loss $W_{\text{nc}} = -f\Delta s$ reduces the total; compare speeds at equal heights with/without friction.

9.5 Exercises

1. **Units.** What are the SI units of U, K, and E? Verify consistency for $U = \frac{1}{2}kx^2$.
2. **Turning points.** For $U = \frac{1}{2}kx^2$ and total E, find the turning points and the maximum speed.
3. **Gravitational climb.** A mass rises from y_0 to $y_1 > y_0$ without friction. Find v_1 from energy.
4. **Practical: Springs.** Compress a small spring by measured x; estimate stored energy $\frac{1}{2}kx^2$ using a catalog value of k.
5. **Friction loss.** With kinetic friction $f = \mu_k N$, estimate the percentage of initial kinetic energy lost over Δs.

9.6 Summary and Review

A brief checklist of the main ideas from this chapter:

- Conservative forces admit a potential U with $F = -\nabla U$ (or $F_x = -\mathrm{d}U/\mathrm{d}x$ in 1D).
- Energy conservation: $E = K + U$ constant; turning points where $E = U$.
- Energy diagrams allow quick, qualitative predictions of speed and accessible regions.

9.7 Where We're Heading Next

In Chapter 10, we model systems of particles, center of mass, and momentum conservation—powerful ideas for collisions and collective motion.

Common Pitfalls

Keep these in mind:

- Choosing a "zero of U" and then thinking it changes physics—only differences in U matter.
- Assuming E is constant when non-conservative forces (like friction) do work.
- Misreading turning points: $E = U$ marks $v = 0$; motion is forbidden where $E < U$.

Try in 60 seconds

Quick checks with energy diagrams:

- Draw $U = \frac{1}{2}kx^2$ and a horizontal E; mark turning points and where speed is largest.
- Raise U by a constant (change the zero). Does anything measurable change? Explain.
- For $U = mgy$, move from y_0 to y_1 without friction. Write v_1 from E in one line.

Part V

Systems, Momentum, and Rotation

Part V Overview

We step from single particles to systems and spinning bodies. Chapter 10 develops center of mass, momentum, and conservation for collisions. Chapter 11 introduces rotational kinematics and dynamics, relating torques to angular acceleration with energy and momentum analogies.

Chapter 10

Systems of Particles and Momentum

Real objects are made of many parts. Treating them as systems leads to simple, powerful summaries: center of mass (COM) tracks collective position; momentum tracks collective motion and is conserved for isolated systems.

Learning Objectives

You will compute a COM in 1D/2D, write momentum balances for systems, recognize impulse as area under force–time, and use momentum conservation for simple collisions.

Symbols at a Glance

$\boldsymbol{R}$ center of mass, M total mass, $\boldsymbol{r}_i$ position of mass m_i, $\boldsymbol{p} = m\boldsymbol{v}$ momentum, $\sum \boldsymbol{p}$ total momentum, impulse $\boldsymbol{J}$.

Analogy: Crowd and Spokesperson

The COM is the crowd's "spokesperson": where the mass acts as if concentrated. Momentum is the spokesperson's "inertia of motion"—harder to change for heavier/faster crowds.

10.1 Center of Mass

Intuition first: the COM is the balancing point. If you could support the system at a single location without it tipping, that location would be $\boldsymbol{R}$. For point masses in 2D,

$$\boldsymbol{R} = \frac{\sum_i m_i \boldsymbol{r}_i}{\sum_i m_i}, \qquad M = \sum_i m_i.$$

The formula says "average the positions, but weight by masses." For two masses on a line, the COM lies on the segment joining them and sits closer to the heavier mass in the ratio of masses.

Two key dynamical facts follow directly (in any inertial frame): the total momentum equals total mass times COM velocity, and external forces move the COM like a single particle of mass M:

$$\boldsymbol{P} = \sum_i m_i \dot{\boldsymbol{r}}_i = M\,\dot{\boldsymbol{R}}, \qquad M\,\ddot{\boldsymbol{R}} = \sum \boldsymbol{F}_{\text{ext}}.$$

Internal forces cancel in pairs (Newton's third law), so they do not affect $\ddot{\boldsymbol{R}}$. This is why momentum conservation is so powerful for isolated systems: if $\sum \boldsymbol{F}_{\text{ext}} = \boldsymbol{0}$, then $\dot{\boldsymbol{R}}$ is constant and $\boldsymbol{P}$ is conserved.

As previewed in Figure 10.1, increasing m_2 pulls the COM toward m_2; equal masses place the COM at the midpoint.

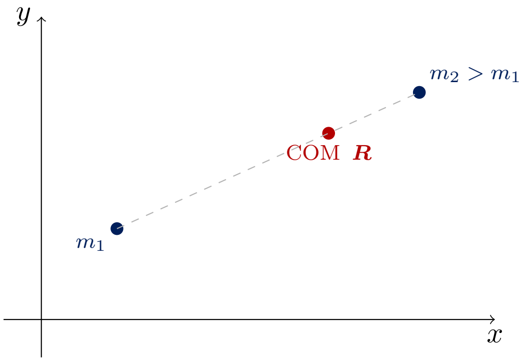


Figure 10.1: Center of mass of two point masses lies along the connector, closer to the heavier mass.

10.2 Momentum and Impulse

Momentum adds across parts: $\sum \boldsymbol{p}$ is the system's "motion budget." External forces change it; internal forces cancel in pairs. Over a short time with a large force (a shove), it is easiest to think in terms of *impulse*

$$\boldsymbol{J} = \int_{t_1}^{t_2} \boldsymbol{F}\,\mathrm{d}t, \qquad \sum \boldsymbol{p}_2 = \sum \boldsymbol{p}_1 + \boldsymbol{J}.$$

As Figure 10.2 shows, the shaded area under $F(t)$ is the impulse. A narrow, tall spike and a short, wide push can deliver the same area and therefore the same change in momentum.

Only *external* impulse changes total momentum: $\Delta \boldsymbol{P} = \boldsymbol{J}_{\text{ext}}$. Equivalently for the COM,

$$\dot{\boldsymbol{R}}_2 = \dot{\boldsymbol{R}}_1 + \frac{\boldsymbol{J}_{\text{ext}}}{M}.$$

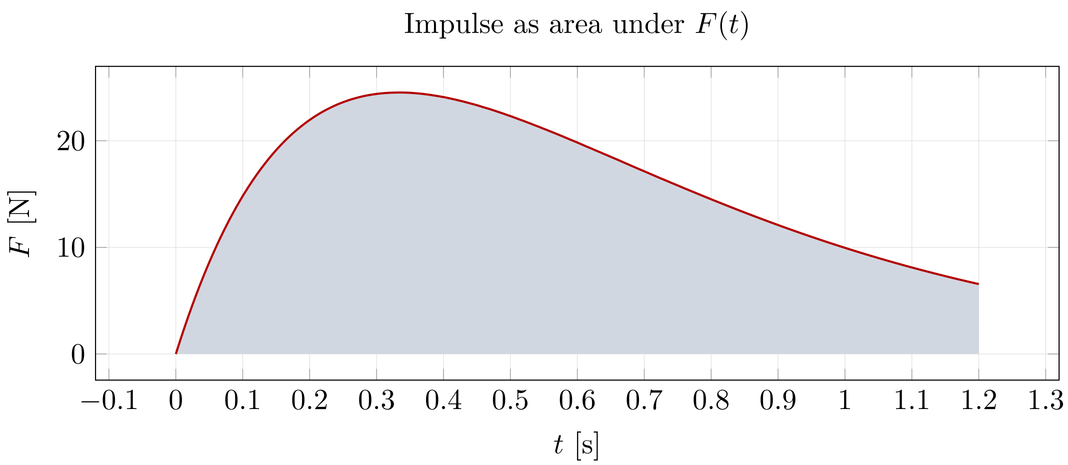


Figure 10.2: A shove: the impulse J is the shaded area under force–time.

Worked Example: Impulse from a Triangular Force Pulse

Using Figure 10.2 as the template, the shaded area under $F(t)$ *is* the impulse J. For an ideal triangular pulse of base Δt and peak F_{max}, $J \approx \frac{1}{2}F_{\text{max}}\,\Delta t$.
A baseball ($m = 0.145\,\text{kg}$) experiences a roughly triangular bat force: rises linearly from 0 to 5000 N over 2.0 ms, then falls back to 0 over the next 2.0 ms. The impulse is that triangle's area

$$J = \tfrac{1}{2}\,F_{\text{max}}\,\Delta t = \tfrac{1}{2}\,(5000)\,(4.0\times 10^{-3})\,\text{N s} = 10.0\,\text{N s}.$$

The speed change is $\Delta v = J/m \approx 69\,\text{m/s}$ along the force direction. Different pulse shapes with the same shaded area in Figure 10.2 deliver the same J and thus the same Δv.

10.3 Collisions

During a brief, isolated impact, external forces are negligible compared to the internal forces between the bodies, so total momentum is conserved:

$$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2 \quad \text{(1D)}.$$

Elastic collisions also conserve kinetic energy; inelastic ones do not (some energy becomes heat/deformation). Conservation of momentum predicts directions and relative magnitudes even when details of the contact are complicated.

In 1D, the *coefficient of restitution* e summarizes how bouncy the collision is via relative speeds:

$$e = \frac{v_2 - v_1}{u_1 - u_2}, \qquad 0 \le e \le 1.$$

Here $e = 1$ is elastic (kinetic energy conserved), and $e = 0$ is perfectly inelastic (stick together). Momentum plus e determines v_1, v_2; use energy conservation only for $e = 1$ (see Chapter 8).

Before Figure 10.4, make a sketch with arrows: heavier objects tend to reverse lighter ones; equal masses exchange speeds in a head-on elastic collision.

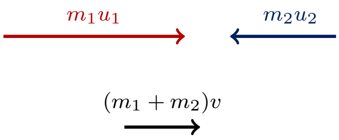


Figure 10.3: Inelastic stick collision: momenta before (top) and combined momentum after (bottom).

Worked Example: Inelastic "Stick" Collision

Cart A ($m_1 = 1.0\,\text{kg}$) moves right at $u_1 = 3.0\,\text{m/s}$. Cart B ($m_2 = 2.0\,\text{kg}$) moves left at $u_2 = -1.0\,\text{m/s}$. They latch (perfectly inelastic). Momentum conservation gives the common speed

$$v = \frac{m_1u_1 + m_2u_2}{m_1 + m_2} = \frac{(1)(3) + (2)(-1)}{3} = \frac{1}{3}\,\text{m/s} \quad \text{(to the right)}.$$

Kinetic energy drops: $K_{\text{before}} = \frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2 = \frac{1}{2}(1)(9) + \frac{1}{2}(2)(1) = 5.5\,\text{J}$; $K_{\text{after}} = \frac{1}{2}(m_1+m_2)v^2 = \frac{1}{2}(3)(1/9) = 0.167\,\text{J}$. The missing $\approx 5.33\,\text{J}$ became heat/deformation. Figure 10.4 sketches before/after momentum arrows and the direction of motion.

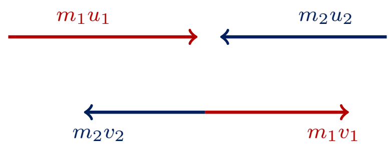


Figure 10.4: 1D collision: momentum arrows before (top) and after (bottom).

10.4 Exercises

Practice reading COM, impulse, and collision statements.

1. **COM of two masses.** Find the COM for m_1 at $x = 0$ and m_2 at $x = L$; where is it when $m_2 = 2m_1$?
2. **Impulse area.** Estimate the area in Figure 10.2 and relate it to a change in momentum.
3. **Collision direction.** Two carts collide head-on with $m_2 > m_1$; argue which way the COM moves during impact.

10.5 Summary and Review

A quick checklist before moving on:

- Center of mass summarizes where mass "acts"; it's a weighted average of positions.
- Momentum adds for subsystems and changes only by external impulse ($\Delta \boldsymbol{P} = \boldsymbol{J}_{\text{ext}}$).
- Short impacts: use conservation of momentum for quick, robust predictions.

10.6 Where We're Heading Next

In Chapter 11, we introduce rotation: angular kinematics, torque, and rotational energy.

Common Pitfalls

Short reminders to avoid mistakes:

- Mixing up internal and external forces in momentum balances—only externals change $\sum \boldsymbol{p}$.
- Forgetting to declare axis directions and signs on diagrams.
- Using energy conservation in inelastic collisions—momentum is conserved, kinetic energy generally is not.

Try in 60 seconds

Fast practice:

- Place two coins on a line and guess their COM; check by balancing a ruler.
- Sketch a force pulse and mark the impulse as area.
- Draw before/after momentum arrows for a gentle bounce vs. a stick.

Chapter 11

Rotation of Rigid Bodies

Rotation extends motion to angles and torques. We mirror linear ideas with angular ones and preview energy and momentum in rotational form.

Learning Objectives

You will describe angular position/velocity/acceleration, relate torque to angular acceleration, and recognize rotational kinetic energy.

Symbols at a Glance

θ angle, ω angular velocity, α angular acceleration, τ torque, r lever arm, I moment of inertia (preview).

Analogy: Door and Handle

Pushing a door near the hinge changes little; at the handle it turns easily. Torque combines force and lever arm—how hard and how far from the pivot.

11.1 Angular Kinematics

Angular position θ measures "how far turned," $\omega = \dot{\theta}$ how fast it turns, and $\alpha = \dot{\omega}$ how quickly the turn rate changes. These mirror linear kinematics: angle plays the role of position, angular velocity that of speed, and angular acceleration that of linear acceleration. At constant α,

$$\omega(t) = \omega_0 + \alpha t, \qquad \theta(t) = \theta_0 + \omega_0 t + \tfrac{1}{2}\alpha t^2.$$

Reading a plot of $\theta(t)$ or $\omega(t)$ is like reading $x(t)$ and $v(t)$ from Chapter 3: slopes give rates; areas give accumulated angle.

Use radians for calculus with angles. For motion along a circle of radius R, arc length and angle relate by $s = R\theta$, so $v = \dot{s} = R\omega$ and $a_{\mathrm{t}} = R\alpha$; the inward (normal) part remains $a_{\mathrm{n}} = v^2/R$ (Chapter 6).

11.2 Torque and Lever Arm

A force $\boldsymbol{F}$ at position vector $\boldsymbol{r}$ produces torque $\boldsymbol{\tau} = \boldsymbol{r} \times \boldsymbol{F}$ about the origin. In the plane, $\tau = rF\sin\phi$ (scalar out of page), where ϕ is the angle between $\boldsymbol{r}$ and $\boldsymbol{F}$. Long lever arms amplify torque; perpendicular pushes are most effective. Right-hand rule: curl fingers from $\boldsymbol{r}$ toward $\boldsymbol{F}$—the thumb points along $\boldsymbol{\tau}$.

In rotational dynamics, torque is to angular acceleration what force is to linear acceleration. With moment of inertia I (a measure of rotational inertia), the parallel of Newton's second law is

$$\tau_{\text{net}} = I\,\alpha \qquad \text{(about a fixed axis; preview).}$$

The rotational kinetic energy analogue is $K_{\text{rot}} = \frac{1}{2}I\omega^2$.

Work and power have clean rotational forms (compare Chapter 8): an infinitesimal angular displacement $\mathrm{d}\theta$ under torque τ does work $\mathrm{d}W = \tau\,\mathrm{d}\theta$, and the instantaneous power is $P = \tau\,\omega$.

Before Figure 11.1, note: longer lever arms and perpendicular pushes maximize torque.

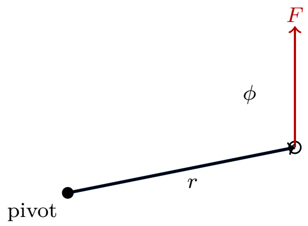


Figure 11.1: Torque about a pivot: $\tau = rF\sin\phi$ measures how strongly $\boldsymbol{F}$ twists around the pivot.

Worked Example: Wrench and Angle

Torque depends on the *perpendicular* component of the pull. With wrench length r and pull F at angle ϕ to the wrench, only $F_\perp = F\sin\phi$ twists, so $\tau = rF\sin\phi$. For $r = 0.25\,\text{m}$ and $F = 120\,\text{N}$: a perpendicular pull ($\phi = 90°$) yields $\tau = 30\,\text{N m}$; at $\phi = 30°$, $\tau = 15\,\text{N m}$. Doubling to $r = 0.50\,\text{m}$ at 30° restores $\tau = 30\,\text{N m}$. See Figure 11.2 for the geometry and Figure 11.3 for how τ varies with angle and lever length.

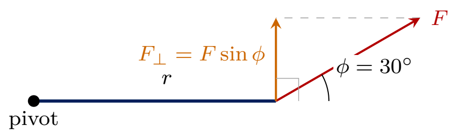


Figure 11.2: Only the perpendicular component $F_\perp$ creates torque about the pivot; $\tau = rF\sin\phi$.

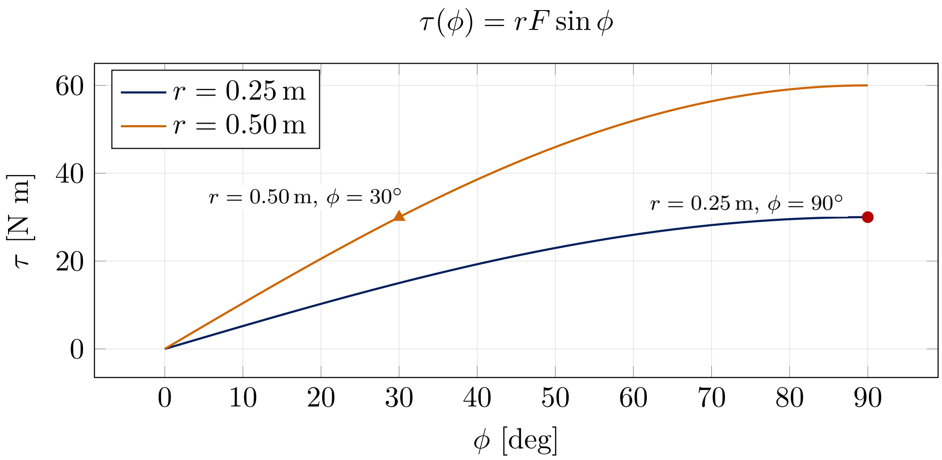


Figure 11.3: Torque vs. angle for two wrench lengths. Longer wrenches lift the $rF\sin\phi$ curve; note how $r = 0.50\,\text{m}$ at 30° matches the perpendicular pull on the shorter wrench.

11.3 Moment of Inertia and Angular Momentum

For rotation about a fixed axis, the *moment of inertia* measures how mass resists angular acceleration:

$$I = \sum_i m_i r_{\perp,i}^2 \quad \text{(point masses)}, \qquad I = \int r_\perp^2 \, dm \quad \text{(continuous)}.$$

About an axis parallel to the center-of-mass (COM) axis and offset by distance d, the parallel-axis relation gives $I = I_{\text{COM}} + Md^2$. The *angular momentum* about the axis is $L = I\,\omega$ (for a rigid body with fixed axis), and the torque law can be written as

$$\tau_{\text{ext}} = \frac{dL}{dt} = I\,\alpha \quad \text{(if } I \text{ is constant about the axis).}$$

When $\tau_{\text{ext}} = 0$, angular momentum is conserved. Rotational kinetic energy mirrors the linear form: $K_{\text{rot}} = \frac{1}{2}I\omega^2$.

11.4 Torque vs. Horsepower in Cars

Torque τ is a twisting tendency (how hard the engine turns the crank). Horsepower is *power*—how fast work is done. The bridge is one line:

$$\boxed{P = \tau\,\omega}$$

where ω is the crankshaft's angular speed (in rad/s). At the same torque, revving faster multiplies power; at the same speed, more torque multiplies power.

Before Figure 11.4, keep an eye on shapes: typical street engines have a broad torque "mesa" in the mid-range; power rises with RPM because $P = \tau\,\omega$.

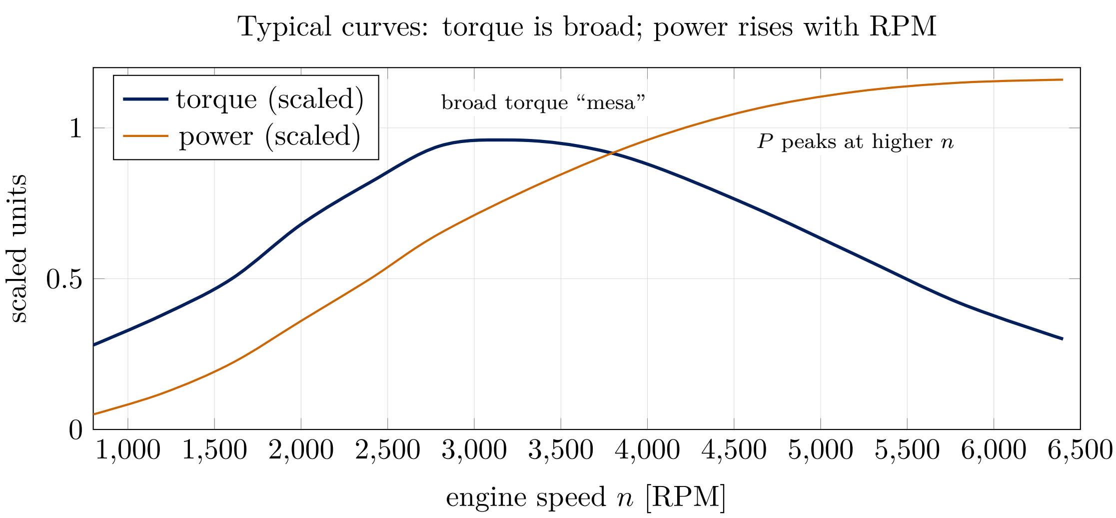


Figure 11.4: A stylized engine map: torque is broad in the mid-range; power grows with n because $\omega = 2\pi n/60$.

What You Feel vs. How Fast You Go

At low speed in a short gear, wheel torque (engine torque multiplied by gear and final drive ratios, minus losses) gives strong *acceleration feel*. At high speed, aerodynamic drag grows rapidly and you need *power* to hold speed—hence peak power roughly sets top speed (all else equal). Gearing trades RPM for wheel torque: lower gears multiply engine torque more, higher gears trade torque for speed.

Worked Example: Back-of-the-Envelope

An engine rated at 150 HP at 6000 RPM. Convert to torque at that RPM. Using $1\,\text{HP} \approx 746\,\text{W}$, $P \approx 112\,\text{kW}$, and $\omega = 2\pi n/60 \approx 2\pi \cdot 6000/60 \approx 628\,\text{rad/s}$,

$$\tau = \frac{P}{\omega} \approx \frac{112 \times 10^3}{628} \approx 178\,\text{N m}.$$

If the same engine makes $\tau \approx 240\,\text{N m}$ at 3000 RPM, then $P = \tau\,\omega \approx 240 \cdot (2\pi \cdot 3000/60) \approx 75\,\text{kW}$—lower power at lower RPM despite higher torque.

Analogy: Levers and Ladders

Gears are levers for rotation. A short ladder (low gear) lifts you quickly over small vertical distances (strong push off the line). A long ladder (high gear) covers more distance per step but each step changes height less (less wheel torque, more road speed).

For a power-based view and efficiency notes, revisit Chapter 8. The equation $P = \boldsymbol{F} \cdot \boldsymbol{v}$ at the wheels ties road force and speed to the engine's $P = \tau\,\omega$ through gearing and losses.

Worked Example: Rolling Without Slipping

A bicycle wheel of radius $R = 0.34\,\text{m}$ rolls without slipping at $v = 6.0\,\text{m/s}$. The angular speed is $\omega = v/R \approx 17.6\,\text{rad/s}$. Points at the rim have velocity $\boldsymbol{v}_{\text{rim}} = \boldsymbol{v}_{\text{CM}} + \boldsymbol{\omega} \times \boldsymbol{r}$: the top point moves at $2v$ relative to the ground and the bottom point momentarily at 0. Figure 11.5 shows the velocity arrows.

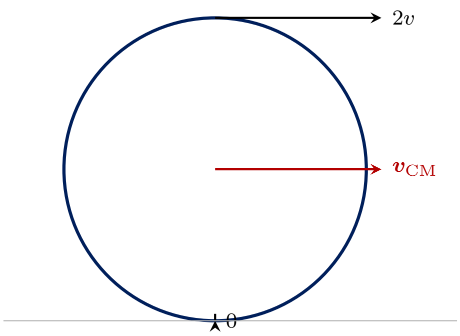


Figure 11.5: Rolling without slipping: $v = \omega R$. Top point instant speed $= 2v$; bottom point instant speed $= 0$.

11.5 Exercises

Practice reading angular motion and torque pictures.

1. **Turn rate.** Convert 60 RPM to rad/s.

2. **Best push.** On a wrench of length r, at what angle do you push for maximum torque?

3. **Practical: Door test.** Try the door near hinge vs. handle; describe the torque difference.

11.6 Summary and Review

A quick checklist of angular ideas:

- Angular kinematics parallels linear: θ, $\omega = \dot{\theta}$, $\alpha = \dot{\omega}$.
- Torque combines force and lever arm: $\tau = \boldsymbol{r} \times \boldsymbol{F}$; perpendicular force maximizes the twist.
- Rotational dynamics preview: $\tau_{\text{net}} = I\,\alpha$, $K_{\text{rot}} = \frac{1}{2}I\omega^2$.

11.7 Where We're Heading Next

In Chapter 12 we turn to Newtonian gravitation—fields, potential, and simple orbits—then return to oscillations and fluids with this rotational toolkit in mind.

Common Pitfalls

Avoid these frequent slips:

- Calling any force a "torque" without specifying the pivot and lever arm.
- Pushing along the lever instead of perpendicular to it—yields tiny torque.
- Mixing radians with degrees when using ω and α.

Try in 60 seconds

Quick angular drills:

- Point at a rotating object (fan/wheel) and identify θ, ω, α qualitatively.
- On Figure 11.1, explain why pushing perpendicular to $\boldsymbol{r}$ maximizes torque.
- Convert 120 RPM to rad/s.

Part VI

Gravity, Oscillations, and Continuum Basics

Part VI Overview

We widen the scope to gravity, oscillations, and simple continuum ideas. Chapter 12 develops Newtonian gravitation and gravitational potential; Chapter 13 studies simple harmonic motion and damping; Chapter 14 sketches essentials of fluids within the Newtonian framework.

Chapter 12

Newtonian Gravitation

Gravity connects motion on Earth with motion of the planets. In Newton's picture, masses attract with a force directed along the line joining them and proportional to the product of masses and inversely to the square of their separation. The same rule that pulls an apple downward bends the Moon's path into a near circle—one idea, many scales.

Learning Objectives

You will write and interpret Newton's law of gravitation, relate gravitational field and potential, and reason about simple orbit conditions.

Symbols at a Glance

G gravitational constant, m, M masses, r separation, $\boldsymbol{g}$ gravitational field, U gravitational potential energy.

Analogy: Invisible Springs

Think of gravity as an invisible spring that always pulls along the line joining two masses—weak when far apart, stronger when closer (inverse-square rather than Hooke's linear law).

12.1 Newton's Law of Universal Gravitation

Start with the rule itself, then read it in pictures. For two point masses m and M separated by distance r, the magnitude of the gravitational force is For clarity we denote this distance as $R_{\oplus M}$ (rather than using punctuation in the subscript).

$$F = G\,\frac{mM}{r^2}, \qquad \text{directed along the line joining the masses.}$$

In vector form (on m due to M):

$$\boldsymbol{F}(\boldsymbol{r}) = -\,G\,\frac{mM}{r^2}\,\hat{\boldsymbol{r}} = -\,\nabla\big(-\tfrac{GmM}{r}\big), \qquad \hat{\boldsymbol{r}} = \frac{\boldsymbol{r}}{\|\boldsymbol{r}\|}.$$

Superposition holds: fields and forces from multiple sources add as vectors. Because gravity is central (along $\hat{\boldsymbol{r}}$), angular momentum about the attracting mass is conserved so motion lies in a plane.

We summarize the "effect per kilogram" with the gravitational field of M:

$$\boldsymbol{g}(\boldsymbol{r}) = -\,GM\,\frac{\hat{\boldsymbol{r}}}{r^2} \qquad \text{(points inward, weakens with } 1/r^2\text{).}$$

Before Figure 12.1, note how field arrows get shorter (weaker) as you move away.

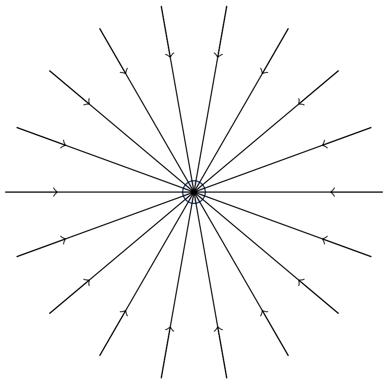


Figure 12.1: Radial gravitational field around a point mass: arrows point inward and weaken with distance.

Notation Used Below

We write $R_\oplus$ for Earth's radius, $M_\oplus$ for Earth's mass, and $R_{\oplus M}$ for the Earth–Moon center-to-center distance. The near-Earth gravitational acceleration is $g \approx 9.81\,\mathrm{m/s^2}$, and AU denotes the astronomical unit.

12.2 Potential and Energy

For conservative forces, energy helps us reason quickly. Choosing $U(\infty) = 0$, the potential energy of m in the field of M is

$$U(r) = -\frac{GmM}{r}, \qquad \Phi(r) = \frac{U(r)}{m} = -\frac{GM}{r}.$$

Two features stand out: U is negative for bound states (you must do work to escape), and $U \to 0$ as $r \to \infty$. Forces and potentials are linked by $\boldsymbol{F} = -\nabla U$ (Appendix Appendix C). As shown in Figure 12.2, the graph dives downward and creeps toward zero far away.

It is convenient to define the gravitational parameter $\mu = GM$. The specific mechanical energy (per unit mass) is

$$\varepsilon = \frac{v^2}{2} + \Phi(r) = \frac{v^2}{2} - \frac{\mu}{r}.$$

Bound orbits have $\varepsilon < 0$ (ellipses), escape trajectories have $\varepsilon \geq 0$ (parabolic at 0, hyperbolic if > 0). For a circular orbit of radius R, $\varepsilon = -\mu/(2R)$, since $v^2 = \mu/R$.

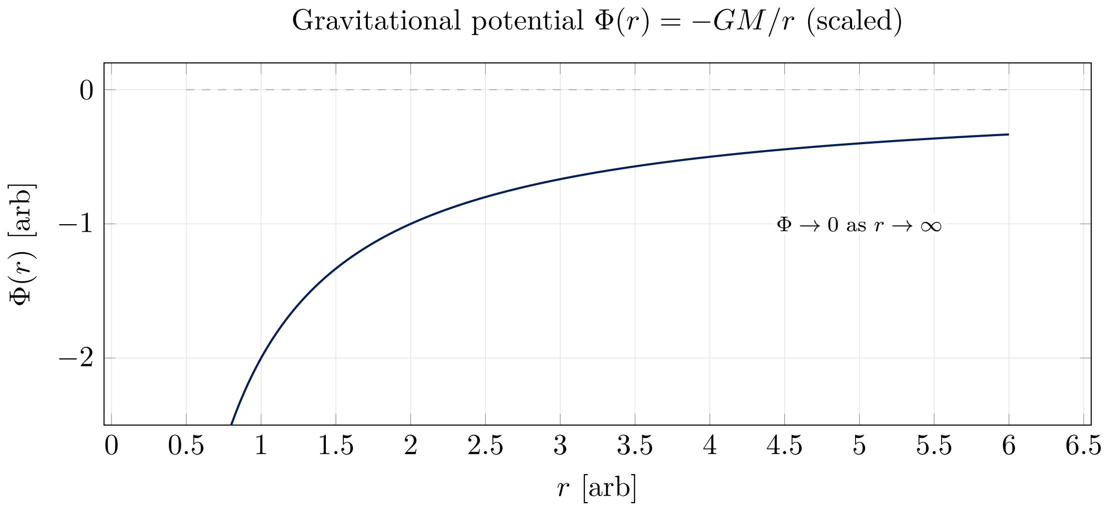


Figure 12.2: Gravitational potential decreases in magnitude with distance and approaches zero from below.

12.3 Circular Orbit Condition

An object of mass m orbits M in a circle of radius R when gravity supplies precisely the needed centripetal acceleration. Equate inward forces:

$$\frac{mv^2}{R} = G\,\frac{mM}{R^2} \quad\Rightarrow\quad v(R) = \sqrt{\frac{GM}{R}}, \qquad T = \frac{2\pi R}{v} = 2\pi\sqrt{\frac{R^3}{GM}}.$$

As shown in Figure 12.3, orbit speed decreases with altitude like $R^{-1/2}$, while the period grows like $R^{3/2}$ (Kepler's third law for circular orbits). Near Earth's surface, $g = \mu/R_\oplus^2$ connects μ to the familiar g.

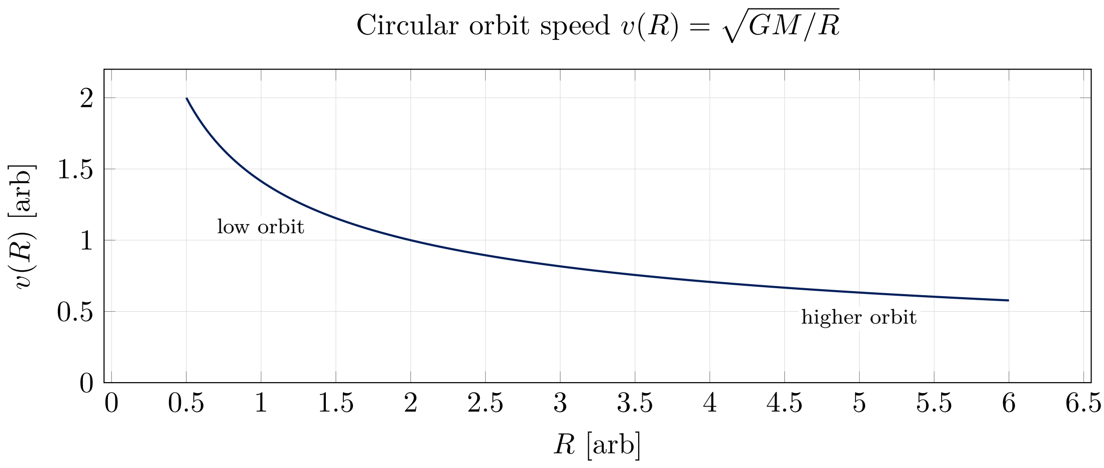


Figure 12.3: Circular orbit speed falls as $R^{-1/2}$: farther means slower.

An energetic bonus: the *escape speed* from radius R follows from energy conservation with $E = 0$ at infinity: $\frac{1}{2}mv_{\text{esc}}^2 - GMm/R = 0$, so $v_{\text{esc}} = \sqrt{\frac{2GM}{R}} = \sqrt{2}\,v_{\text{circ}}$.

Worked Example: Parking a Geostationary Satellite

To remain above one longitude, a satellite must share Earth's sidereal rotation period $T_{\text{sid}} = 86\,164\,\text{s}$. With $\mu_\oplus = GM_\oplus \approx 3.986 \times 10^{14}\,\text{m}^3/\text{s}^2$,

$$R = \left(\frac{\mu_\oplus T_{\text{sid}}^2}{4\pi^2}\right)^{1/3} \approx 4.216 \times 10^7\,\text{m}.$$

Subtract Earth's mean radius $R_\oplus \approx 6.371 \times 10^6\,\text{m}$ to find the altitude: $h = R - R_\oplus \approx 3.58 \times 10^7\,\text{m}$ (about 35 800 km). The orbital speed follows from $v = \sqrt{\mu_\oplus/R} \approx 3.07 \times 10^3\,\text{m/s}$. The geometry is sketched in Figure 12.4.

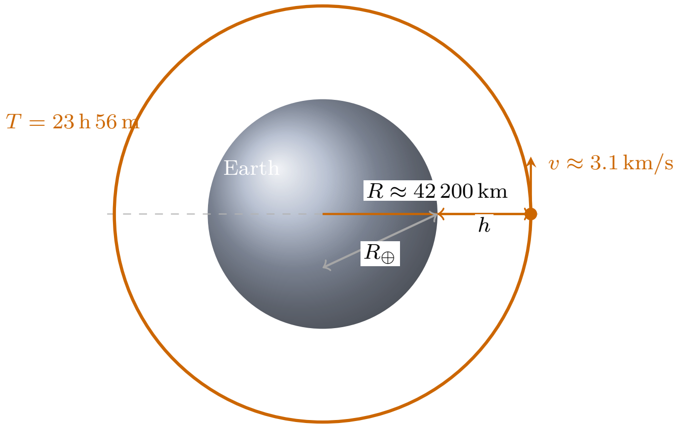


Figure 12.4: Geostationary orbit: the radius $R \approx 42\,200\,\text{km}$ (altitude $h \approx 35\,800\,\text{km}$) yields a sidereal rotation period.

12.4 Everyday Gravity Examples

Gravity's inverse-square law explains apples and orbits with the same symbols. Here are concrete, number-driven examples with a few friendly plots.

Worked Example: Falling Apple

An apple drops from a branch at height $h_0 = 3.0\,\text{m}$ above the ground. Near Earth's surface the gravitational field is nearly uniform, $g \approx 9.81\,\text{m/s}^2$ downward. With zero initial speed,

$$y(t) = h_0 - \tfrac{1}{2}gt^2, \quad v(t) = \dot{y}(t) = -gt.$$

The fall time solves $y(T) = 0$: $T = \sqrt{\frac{2h_0}{g}} \approx \sqrt{\frac{6.0}{9.81}} \approx 0.78\,\text{s}$. The impact speed is $|v(T)| = gT \approx 7.7\,\text{m/s}$ (about 28 km/h).
Air drag would reduce the speed a bit, but over 3 m the constant-g model is an excellent first estimate.

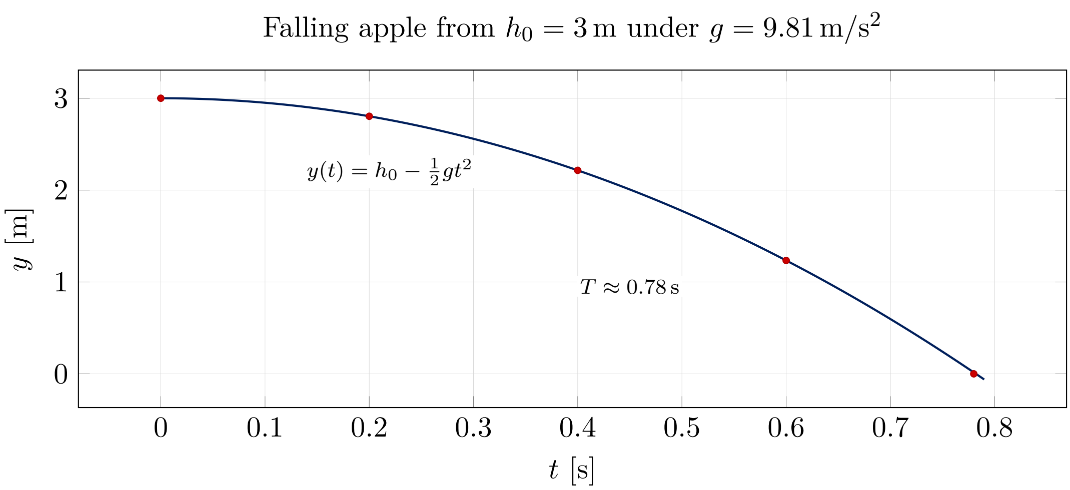


Figure 12.5: Apple height vs. time for a 3 m drop. Dots emphasize discrete time samples; the curve is the constant-g analytic solution.

Moon Held by Earth's Gravity

The Moon's nearly circular path is continuous free fall around Earth. Using $M_\oplus$ and Earth–Moon distance $R_{\oplus ext\text{–Moon}}$,

$$\underbrace{\frac{v^2}{R_{\oplus ext\text{–Moon}}}}_{\text{needed centripetal}} = \underbrace{\frac{GM_\oplus}{R^2_{\oplus ext\text{–Moon}}}}_{\text{gravity from Earth}} \Rightarrow v = \sqrt{\frac{GM_\oplus}{R_{\oplus ext\text{–Moon}}}}.$$

Numerically, insert $GM_\oplus \approx 3.986 \times 10^{14}\,\text{m}^3/\text{s}^2$ and $R_{\oplus M} \approx 3.844 \times 10^8\,\text{m}$:

$$\begin{aligned} v &\approx 1.02 \times 10^3\,\text{m/s}, \\ a &\approx 2.7 \times 10^{-3}\,\text{m/s}^2, \\ T &\approx 2.37 \times 10^6\,\text{s}\ (\approx 27.5\,\text{days}). \end{aligned}$$

Compare $a_{\text{Moon}} \approx 2.7 \times 10^{-3}\,\text{m/s}^2$ to $g \approx 9.81\,\text{m/s}^2$ at Earth's surface: the same law, different distance scale.

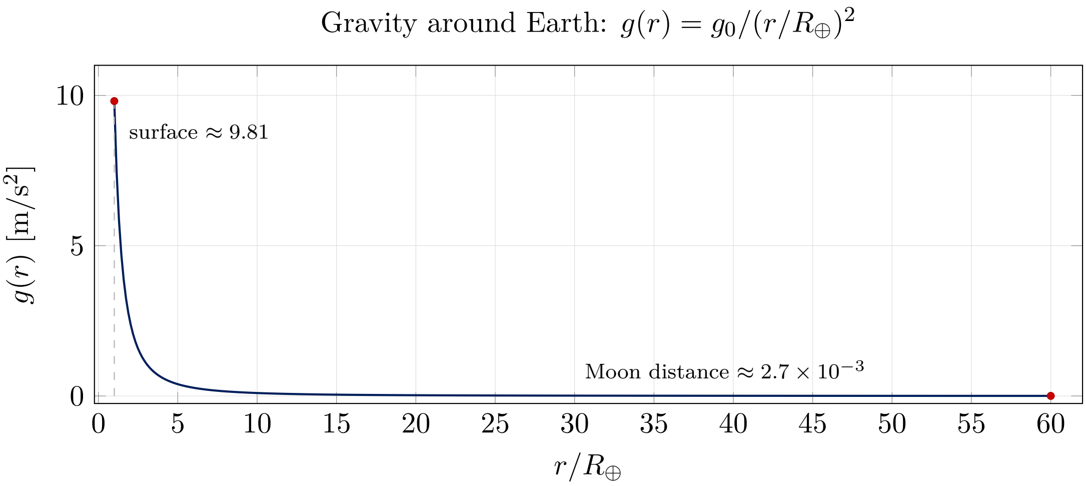


Figure 12.6: How Earth's gravitational acceleration weakens with distance. The Moon orbits where g is about $0.003\,\mathrm{m/s^2}$.

Sun–Earth Numbers, Same Rule

At Earth's orbital radius ($R \approx 1\,\mathrm{AU} = 1.496 \times 10^{11}\,\mathrm{m}$), the Sun's gravity provides

$$\frac{GM_\odot}{R^2} \approx 5.93 \times 10^{-3}\,\mathrm{m/s^2}, \qquad v_\oplus = \sqrt{\frac{GM_\odot}{R}} \approx 29.8\,\mathrm{km/s}.$$

This acceleration is far smaller than g at Earth's surface but acts continuously over vast distances, curving Earth's path into a year-long orbit. One idea links falling fruit and planetary motion.

12.5 Exercises

Practice reading gravity graphs and quick estimates.

1. **Field magnitude.** Compute $\|\boldsymbol{g}\|$ at Earth's surface from $M_\oplus, R_\oplus$; compare with $9.8\,\mathrm{m/s^2}$.
2. **Orbit speed.** Find the speed for a circular orbit at altitude h above a planet of radius R (assume $h \ll R$ for an easy estimate).
3. **Escape vs. circular.** Show that $v_{\mathrm{esc}} = \sqrt{2}\,v_{\mathrm{circ}}$ at the same R.

12.6 Summary and Review

- Inverse-square attraction: $F = GmM/r^2$, field inward with $1/r^2$.
- Potential energy: $U = -GmM/r$ with $U(\infty) = 0$; $\boldsymbol{F} = -\nabla U$.
- Circular orbit: $v = \sqrt{GM/R}$, $T = 2\pi\sqrt{R^3/(GM)}$; specific energy $\varepsilon = -GM/(2R)$.

12.7 Where We're Heading Next

In Chapter 13, we study simple harmonic motion and damping.

Common Pitfalls

Quick cautions: confusing $1/r^2$ with $1/r$; forgetting that U is negative for bound states; assuming circular orbits are the only possibility.

Try in 60 seconds

Quick wins:

- On Figure 12.2, point to where the object is most bound and explain why.
- Read off from Figure 12.3: does doubling altitude more than halve the orbit speed?
- Write the escape speed in terms of g and R for a spherical planet.

Chapter 13

Simple Harmonic Motion and Oscillations

Oscillations are everywhere: springs, pendulums, violin strings. The simplest model is the simple harmonic oscillator (SHO), where the restoring force pulls back toward equilibrium in proportion to displacement.

Learning Objectives

You will derive and solve the SHO equation, interpret $x(t)$, $v(t)$, and the phase portrait, and reason about how energy shuttles between U and K. You will also read the basic effects of damping and driving.

Symbols at a Glance

k spring constant, m mass, $\omega_0 = \sqrt{k/m}$ natural frequency, A amplitude, ϕ phase, γ damping factor (preview), F_0 drive amplitude (preview).

Analogy: Energy Pendulum

Imagine energy sloshing back and forth—when x is largest, energy sits mostly in U; when passing through $x = 0$, energy is mostly in K. Perfectly elastic motion keeps the total full; damping slowly leaks it away.

13.1 From Force Law to Equation

Hooke's law states $F = -kx$. Newton's second law gives

$$m\ddot{x} = -kx \quad \Rightarrow \quad \ddot{x} + \omega_0^2 x = 0, \qquad \omega_0 = \sqrt{\frac{k}{m}}.$$

This constant-coefficient ODE has sinusoidal solutions

$$x(t) = A\cos(\omega_0 t + \phi), \qquad v(t) = \dot{x}(t) = -A\,\omega_0 \sin(\omega_0 t + \phi),$$

with period $T = 2\pi/\omega_0$. As previewed in Figure 13.1, $x(t)$ oscillates smoothly with fixed amplitude and period.

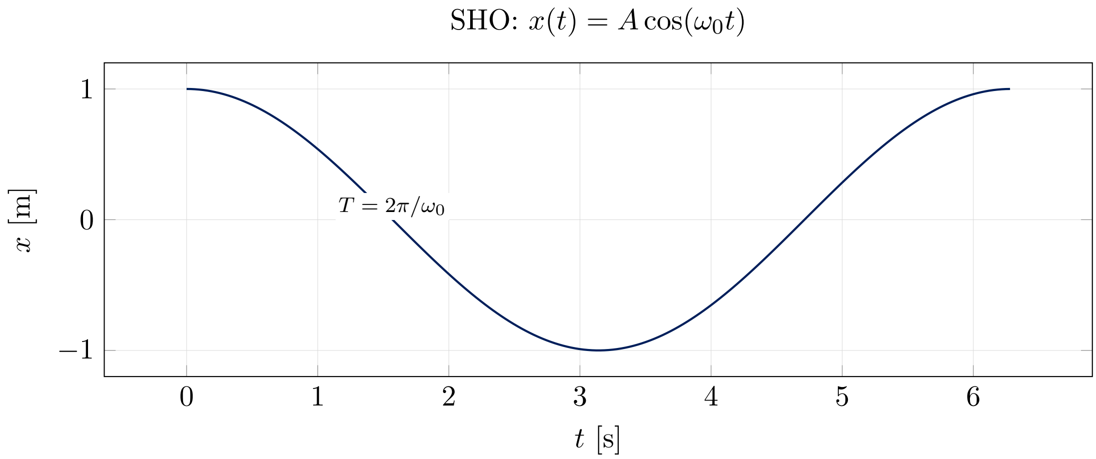


Figure 13.1: Position of an ideal SHO with $A = 1$, $\omega_0 = 1$ (scaled units).

13.2 Phase Portrait and Geometry

Plotting v against x eliminates time and reveals the geometry of SHO: combining $x = A\cos\theta$ and $v = -A\omega_0 \sin\theta$ gives

$$\left(\frac{x}{A}\right)^2 + \left(\frac{v}{A\omega_0}\right)^2 = 1,$$

an ellipse (a circle if axes are scaled equally). Motion runs around the ellipse at constant angular speed in phase space. Figure 13.2 shows the picture.

Energy gives the same ellipse directly. Since $E = \frac{1}{2}kA^2 = \frac{1}{2}kx^2 + \frac{1}{2}mv^2$, divide by $\frac{1}{2}kA^2$ to obtain

$$\frac{x^2}{A^2} + \frac{v^2}{(A\,\omega_0)^2} = 1, \qquad \omega_0 = \sqrt{\frac{k}{m}},$$

which is the phase-space ellipse without invoking sines and cosines. Either route—explicit solution or energy—leads to the same geometry.

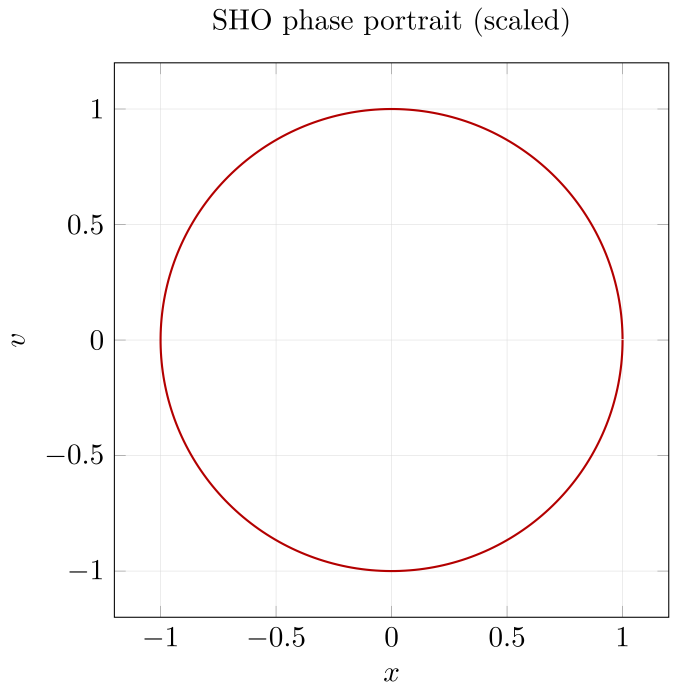


Figure 13.2: Phase portrait for SHO in scaled units: a circle traced at constant speed.

13.3 Energy Picture

The total energy is constant and splits between potential and kinetic:

$$E = \tfrac{1}{2}kx^2 + \tfrac{1}{2}mv^2 = \tfrac{1}{2}kA^2 = \text{constant}.$$

At turning points ($x = \pm A$) energy is all potential; at $x = 0$ energy is all kinetic. Figure 13.3 shades U and K over time.

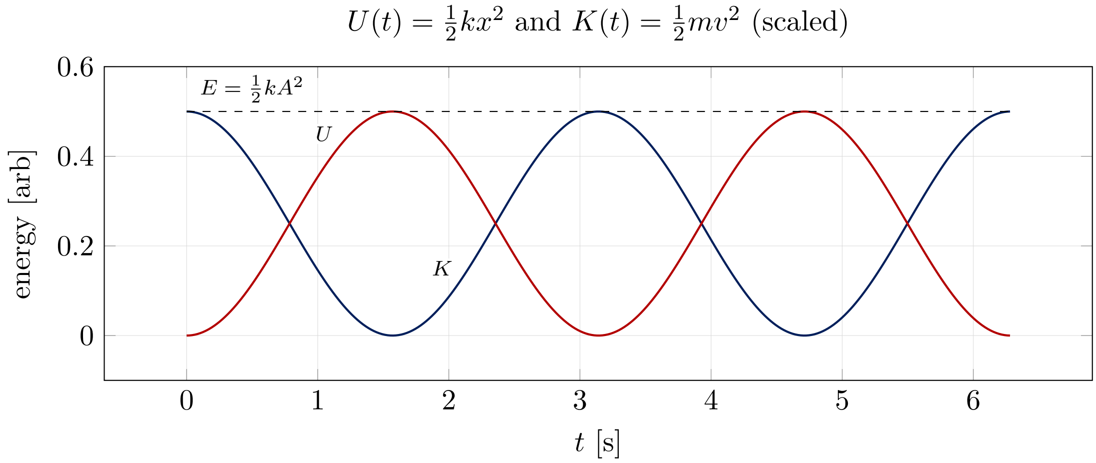


Figure 13.3: Energy exchanges while E stays constant (scaled units: $k = m = A = \omega_0 = 1$).

13.4 Small-Angle Pendulum ≈ SHO

Gravity plus geometry creates another SHO. A simple pendulum of length L and small angle θ from the vertical obeys

$$\text{(tangential)} \quad mL\ddot{\theta} = -mg\sin\theta \;\approx\; -mg\,\theta \quad (|\theta| \ll 1),$$

so $\ddot{\theta} + (g/L)\,\theta = 0$ with natural frequency $\omega_0 = \sqrt{g/L}$. Thus the small-angle pendulum is an SHO in θ. This links back to Chapter 12: gravity supplies the restoring "spring." The linearization $\sin\theta \approx \theta$ is accurate for modest angles (say $|\theta| \lesssim 10°\text{–}15°$); beyond that, the period grows slightly with amplitude.

Before Figure 13.4, note that the restoring component is $mg\sin\theta$ along the tangent and that arc length $s \approx L\theta$ for small angles.

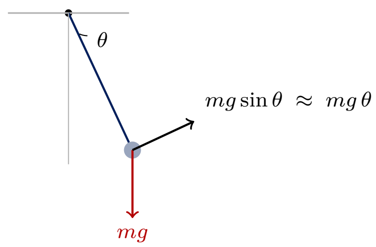


Figure 13.4: Small-angle pendulum: the tangential restoring component $mg\sin\theta \approx mg\,\theta$ leads to the SHO equation in θ.

13.5 Light Damping (Preview)

With small resistive forces $F_{\text{damp}} = -b\,v$, the equation becomes $m\ddot{x} + b\dot{x} + kx = 0$. Solutions oscillate with slowly decaying amplitude $A\,e^{-\gamma t}$ where $\gamma = \frac{b}{2m}$ and $\omega \approx \sqrt{\omega_0^2 - \gamma^2}$. It is common to define the damping ratio $\zeta = \dfrac{b}{2\sqrt{mk}} = \dfrac{\gamma}{\omega_0}$ and the quality factor $Q = \dfrac{1}{2\zeta}$ (how many radians of oscillation per e-fold decay). Figure 13.5 shows the envelope.

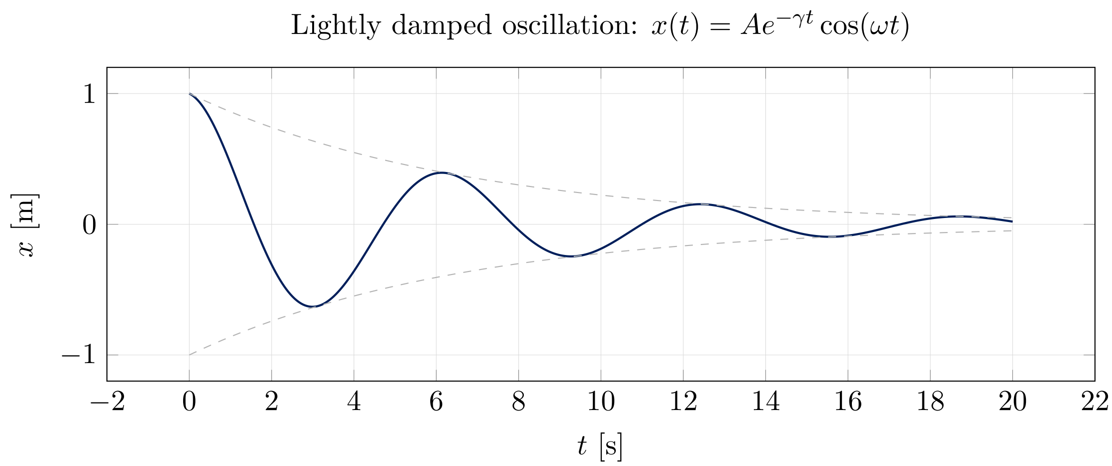


Figure 13.5: Damping shrinks amplitude with an exponential envelope; frequency shifts slightly from ω_0.

Worked Example: Car Suspension over a Speed Bump

A 1 200 kg compact car sits on a strut with spring rate $k = 20\,\mathrm{kN/m}$ and damping $c = 1.5\,\mathrm{kN\,s/m}$. Approximating a 5 cm speed bump as an initial compression $x(0) = 0.05\,\mathrm{m}$ with $v(0) = 0$, the equivalent single degree of freedom has

$$\omega_0 = \sqrt{\frac{k}{m}} \approx 4.08\,\mathrm{rad/s}, \qquad \zeta = \frac{c}{2\sqrt{mk}} \approx 0.15.$$

Because $\zeta < 1$, oscillations persist but decay with envelope $Ae^{-\zeta\omega_0 t}$. The damped frequency is $\omega_d = \omega_0\sqrt{1-\zeta^2} \approx 4.03\,\mathrm{rad/s}$, giving a half-cycle time $\pi/\omega_d \approx 0.78\,\mathrm{s}$. The first rebound reaches $|x| \approx 0.05\,e^{-\zeta\omega_0(\pi/\omega_d)} \approx 0.031\,\mathrm{m}$ (3.1 cm), and the next peak falls to about 0.019 m (1.9 cm). The ride smooths out within a few seconds, as sketched in Figure 13.6.

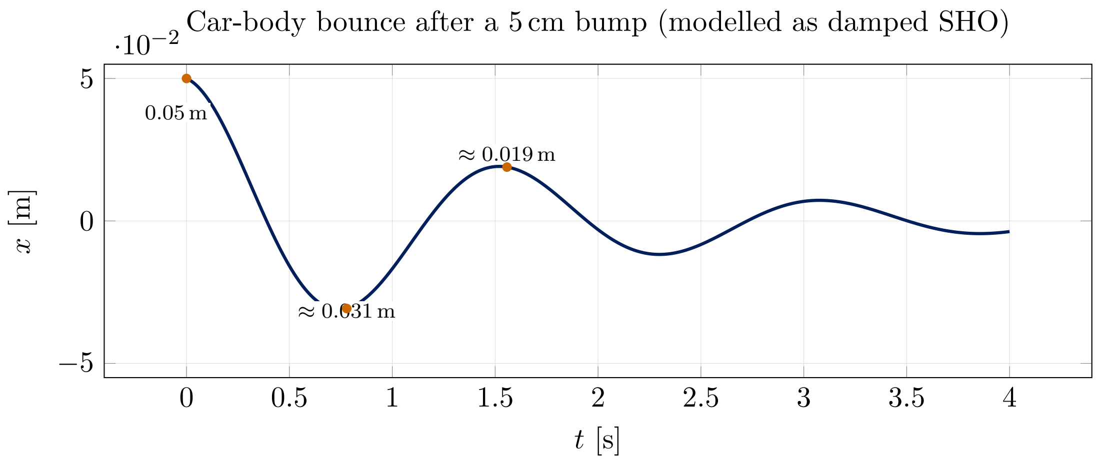


Figure 13.6: Damped bounce response with $m = 1\,200\,\mathrm{kg}$, $k = 20\,\mathrm{kN/m}$, $c = 1.5\,\mathrm{kN\,s/m}$. Peaks shrink quickly as the envelope $e^{-\zeta\omega_0 t}$ decays.

13.6 Driven Response (Preview)

A periodic drive $F_0\cos(\Omega t)$ leads to steady oscillations at the drive frequency Ω with amplitude (for light damping)

$$A(\Omega) = \frac{F_0/m}{\sqrt{(\omega_0^2 - \Omega^2)^2 + (2\gamma\,\Omega)^2}}.$$

Amplitude peaks near $\Omega \approx \omega_0$ (resonance) and broadens with larger γ. The steady-state motion also lags the drive by a phase ϕ with $\tan\phi = \dfrac{2\gamma\,\Omega}{\omega_0^2 - \Omega^2}$. Full derivations live in Appendix Appendix D; Figure 13.7 plots the amplitude shape.

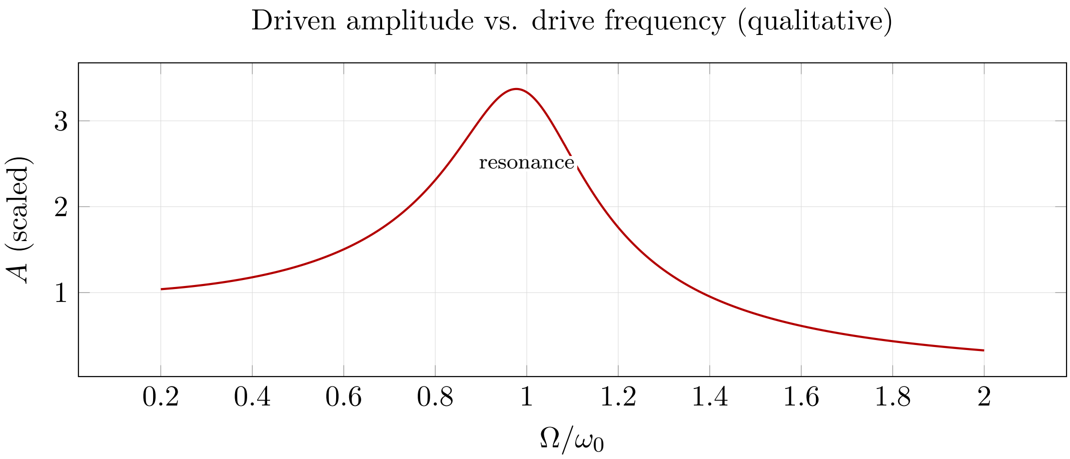


Figure 13.7: Amplitude is largest near the natural frequency; damping broadens and lowers the peak.

13.7 Exercises

Practice deriving, reading, and estimating.

1. **Period.** For $m = 1$, $k = 4$, compute ω_0 and T.
2. **Phase portrait.** Show that $(x/A)^2 + (v/(A\omega_0))^2 = 1$ for the SHO.
3. **Light damping.** If $\gamma = 0.1\,\omega_0$, estimate the amplitude after 10 periods.

13.8 Summary and Review

- SHO: $m\ddot{x} + kx = 0$ with $x(t) = A\cos(\omega_0 t + \phi)$, $\omega_0 = \sqrt{k/m}$.
- Phase-space circle (scaled): motion at constant speed around the ellipse.
- Energy swaps between U and K while E remains constant; damping leaks energy slowly.
- Driving near ω_0 produces large amplitudes (resonance) moderated by damping.

13.9 Where We're Heading Next

In Chapter 14, we sketch how Newtonian ideas extend to simple fluids.

Common Pitfalls

Confusing ω_0 with frequency in Hz ($f = \omega_0/2\pi$); thinking amplitude affects period in the linear SHO; mixing degrees/radians in trig; assuming the phase-space ellipse is axis-aligned without scaling.

Try in 60 seconds

Quick drills:

- From $x(t) = A\cos(\omega_0 t)$, write $v(t)$ and mark a quarter-period phase shift.
- On the phase portrait, point to where U is maximal and where K is maximal.
- If γ doubles, does the envelope decay twice as fast? Explain.

Chapter 14

Fluids and Continuum Mechanics (Introductory)

Many everyday phenomena involve continuous media rather than point masses. We sample Newtonian ideas for fluids: pressure, buoyancy, and a taste of flow—enough to reason about water, air, pipes, and simple lift effects without leaving the Newtonian toolbox.

Learning Objectives

You will reason about hydrostatic pressure, estimate buoyant forces, draw simple fluid free-body diagrams, and read continuity arguments for steady incompressible flow.

Symbols at a Glance

p pressure, ρ density, g gravitational acceleration, A area, Q flow rate ($Q = Av$), h depth.

Analogy: Stacked Bricks

Hydrostatic pressure is like stacked bricks—deeper layers carry more weight. Pressure increases linearly with depth in an incompressible fluid, and differences in pressure across surfaces produce net forces.

14.1 Pressure and Buoyancy

Consider a fluid at rest under gravity. A vertical column of height h carries the weight of the fluid above: balancing forces on a thin slab gives

$$p(h) = p_0 + \rho g h \quad (h \text{ measured downward from the free surface}).$$

Pressure acts equally in all directions at a point. In differential form, hydrostatic balance reads $\nabla p = \rho\,\boldsymbol{g}$; in a uniform $\boldsymbol{g} = -g\,\hat{\boldsymbol{y}}$ this integrates to $p = p_0 + \rho g h$. For a submerged body, the pressure increases with depth on its bottom face relative to its top face, creating a net upward buoyant force equal to the weight of displaced fluid (Archimedes):

$$F_\text{b} = \rho g V_\text{disp}.$$

Float or sink? Compare densities: if $\rho_\text{object} < \rho_\text{fluid}$ the displaced weight can balance the object's weight and it floats; if $\rho_\text{object} > \rho_\text{fluid}$ it sinks. Before Figure 14.1, note the straight-line pressure–depth plot; Figure 14.2 shows a simple buoyancy free-body diagram.

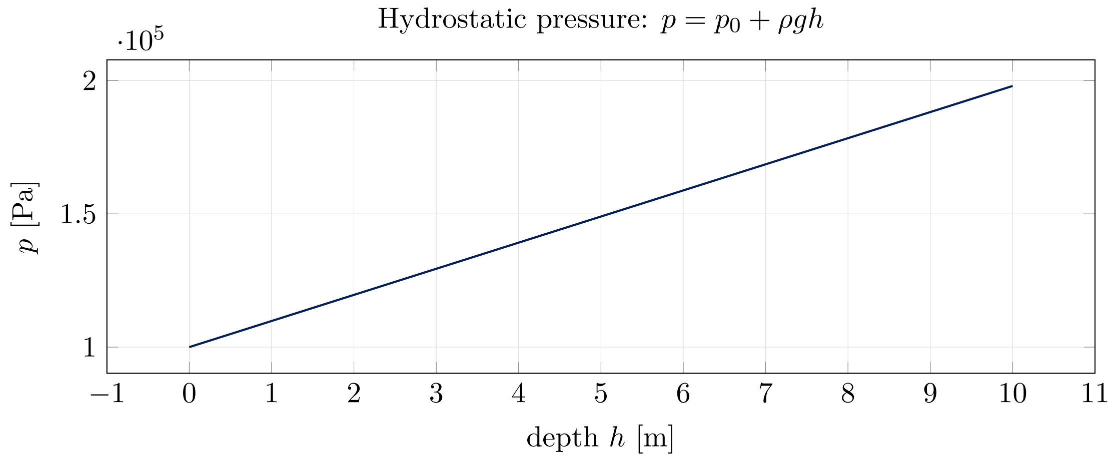


Figure 14.1: Pressure increases linearly with depth in an incompressible fluid.

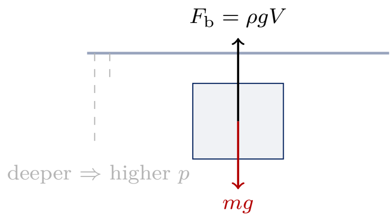


Figure 14.2: Buoyant free-body diagram: pressure increases with depth; the net result is an upward buoyant force that balances some or all of the weight.

14.2 Flow Basics (Preview)

For steady, incompressible flow in a pipe, mass conservation says $\rho A_1 v_1 = \rho A_2 v_2$, i.e. $A_1 v_1 = A_2 v_2$—when the pipe narrows, the fluid speeds up. Along a streamline in slowly varying flow without pumps or strong losses, Bernoulli's principle balances energy per unit volume:

$$p + \frac{1}{2}\rho v^2 + \rho g z \approx \text{constant}.$$

We use it qualitatively to understand tradeoffs between pressure, speed, and height. Assumptions matter: Bernoulli applies along a streamline for inviscid, steady flow (or where viscous losses are small) and incompressible speeds (low Mach). When viscosity or turbulence dominate, expect departures (see Chapter 15 for Reynolds number scaling). Figure 14.3 sketches a narrowing pipe and velocity change. Figure 14.4 visualizes a faster flow over a curved top surface (lower pressure) and a slower flow beneath (higher pressure), producing a net upward force.

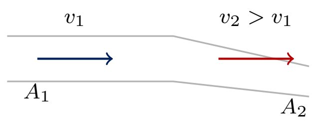


Figure 14.3: Continuity in a narrowing pipe: $A_1 v_1 = A_2 v_2$; reduced area implies higher speed for incompressible steady flow.

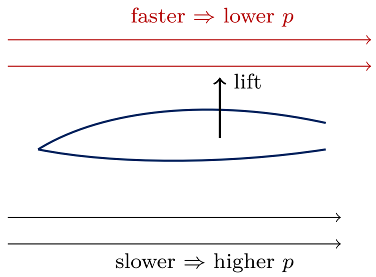


Figure 14.4: Bernoulli lift sketch: faster flow over the curved upper surface reduces pressure relative to the slower flow below, producing net upward force.

Worked Example: Garden Hose Nozzle Jet

A hose of diameter $d_1 = 1.6\,\text{cm}$ feeds a nozzle of diameter $d_2 = 0.5\,\text{cm}$. The pressure just upstream of the contraction is $p_1 = 200\,\text{kPa}$ (gauge) and the jet exits to atmospheric pressure ($p_2 \approx 0$ gauge). Treating water as incompressible with $\rho \approx 1{,}000\,\text{kg/m}^3$ and neglecting height changes,

$$A_1 v_1 = A_2 v_2, \qquad p_1 + \tfrac{1}{2}\rho v_1^2 = p_2 + \tfrac{1}{2}\rho v_2^2.$$

With $A \propto d^2$, $v_2 = (d_1/d_2)^2 v_1 \approx 10.2\,v_1$. Substitute into Bernoulli to solve for v_1:

$$200 \times 10^3 \approx \tfrac{1}{2}\rho\left((10.2^2 - 1)v_1^2\right) \Rightarrow v_1 \approx 2.0\,\text{m/s}, \quad v_2 \approx 20\,\text{m/s}.$$

Tilt the nozzle upward by 30° at waist height ($z \approx 1\,\text{m}$) and ignore drag: the jet range is roughly

$$R \approx \frac{v_2^2}{g}\sin(60^\circ) \approx 31\,\text{m}.$$

Real sprays break into droplets and slow sooner, but the estimate shows how a moderate pressure and narrow nozzle yield a long-reaching jet. See Figure 14.5 for a geometric sketch.

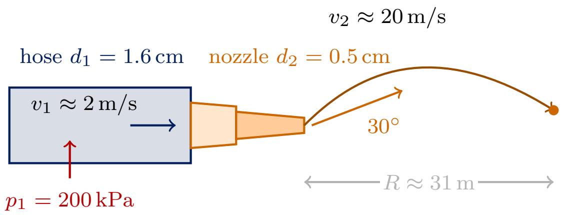


Figure 14.5: Hose-to-nozzle contraction: continuity and Bernoulli give entrance speed $v_1 \approx 2\,\text{m/s}$ and jet speed $v_2 \approx 20\,\text{m/s}$. A 30° launch would reach roughly 30 m neglecting air drag.

14.3 Exercises

Short, concrete checks:

1. **Hydrostatics.** A diver descends 15 m in freshwater ($\rho \approx 1000\,\text{kg/m}^3$). Estimate the gauge pressure increase Δp.

2. **Buoyancy.** A cube of side 0.2 m floats in oil ($\rho \approx 800\,\text{kg/m}^3$) with 3 cm above the surface. Estimate the cube's density.

3. **Continuity.** Water flows steadily through a pipe that narrows from $A_1 = 6\,\mathrm{cm}^2$ to $A_2 = 2\,\mathrm{cm}^2$. If $v_1 = 0.5\,\mathrm{m/s}$, find v_2.

14.4 Summary and Review

- Hydrostatics: $\nabla p = \rho\,\boldsymbol{g}$; $p = p_0 + \rho g h$ in uniform gravity.
- Buoyancy: $F_\mathrm{b} = \rho g V_\mathrm{disp}$; float/sink set by density comparison.
- Steady incompressible flow: continuity $Av = \mathrm{const}$; Bernoulli along a streamline when viscous losses are small.

14.5 Where We're Heading Next

In Chapter 15 we step back and develop general tools—dimensional analysis and scaling—that sharpen intuition and provide quick checks. These ideas pair well with fluids: scaling arguments explain, for example, why small insects can walk on water whereas larger animals cannot.

Common Pitfalls

Assuming Bernoulli without its conditions (inviscid, along a streamline, negligible losses); forgetting that pressure acts in all directions; mixing absolute and gauge pressure; treating compressible high-speed flows as incompressible.

Try in 60 seconds

Quick checks:

- Point to the pressure increase on Figure 14.1 and say why it's linear.
- Use continuity to decide where the fluid speeds up in Figure 14.3.
- State one assumption behind Bernoulli in one sentence.

Part VII

Methods, Modelling, and Numerical Simulation

Part VII Overview

We add cross-cutting tools. Chapter 15 develops dimensional analysis and scaling. Chapter 16 introduces practical numerical methods and modelling workflows that complement analytic solutions.

Chapter 15

Dimensional Analysis and Scaling

Dimensional analysis is a compact way to check equations and predict relationships before calculating. Scaling arguments explain why the same equations look different for small and large systems—why ants "feel" strong and elephants do not.

Learning Objectives

You will check dimensional consistency, build dimensionless groups (π *groups*) from variables and constants, and use scaling to estimate how quantities change with size.

Symbols at a Glance

$[\cdot]$ dimension of a quantity; common base dimensions: M mass, L length, T time. We use $\sim$ for "scales as."

Analogy: Recipe Without Numbers

Dimensional analysis is like a recipe that only lists the *kinds* of ingredients. You can already tell whether the cake is possible before measuring the amounts.

15.1 Consistency Checks

Dimensional homogeneity is the first line of defense: every term you add or compare must represent the same *kind* of quantity. If one side of an equation has the units of force and the other side has the units of energy, the equation cannot be right no matter what numbers you plug in.

Everyday examples help:

- $F = ma$ passes the test because $[ma] = \mathrm{M\,L\,T^{-2}} = [F]$.
- $v = v_0 + at$ is fine: each term has $\mathrm{L\,T^{-1}}$.
- $x = x_0 + at^2$ is fine for constant acceleration, but $x = x_0 + at$ fails (wrong power of time).
- Arguments of $\sin(\cdot)$, $\exp(\cdot)$, $\log(\cdot)$ must be dimensionless. Expressions like $\exp(t)$ are illegal unless t was first non-dimensionalized (e.g., t/τ).

Practical habits that pay off:

- Carry symbolic dimensions through your algebra before substituting numbers.
- Convert mixed measurement systems early (e.g., km/h to m/s) so dimensions remain transparent.

- When in doubt, rewrite a formula to isolate one quantity and check that the isolated quantity has the right dimensions.

Lead-in questions to ask when scanning a formula:

- Do all additive terms share the same dimensions?
- Do arguments of trig/exp/log functions come out dimensionless?

Unit-Check Checklist

Keep this short list handy when reading or writing formulas:

- Left–right match: confirm [LHS] = [RHS].
- Additions: every term you add must share the same dimensions.
- Transcendentals: arguments of $\sin, \cos, \exp, \log$ are dimensionless.
- Constants: numerical constants are dimensionless; if a parameter carries units, track them.
- Mixed systems: convert early (e.g., km/h to m/s) and stick to one system.
- Isolate and check: solve for the target symbol and confirm its dimension.

15.2 Building Dimensionless Groups (by Hand)

Dimensionless groups are combinations of variables that carry no units. They often reveal the natural control knobs of a problem and collapse data taken at different scales onto a single curve.

Worked example (linear drag). Suppose the terminal speed v_t depends on m (mass), g (acceleration), and c (drag coefficient for $F_d = cv$). Write the dimensions

$$[v_t] = \frac{\mathrm{L}}{\mathrm{T}},\ [m] = \mathrm{M},\ [g] = \frac{\mathrm{L}}{\mathrm{T}^2},\ [c] = \frac{\mathrm{M}}{\mathrm{T}}.$$

Seek $v_t \sim m^a g^b c^d$. Matching exponents of $\mathrm{M}, \mathrm{L}, \mathrm{T}$ yields $a = 1,\ b = 1,\ d = -1$, so $v_t \sim mg/c$. We did not need the exact numerical constant to know the form.

For quadratic drag $F_d = \frac{1}{2}\rho C_D A v^2$, momentum exchange with the fluid sets the scale; v_t should grow with weight and shrink with fluid density and cross-section. Try $v_t \sim (mg/\rho A)^{1/2}$, consistent with $[\rho A] = \mathrm{M}/\mathrm{L}$.

Reading data with π groups. If a measured quantity Y depends on $X_1, \ldots, X_n$, form π groups π_i and plot one against another. A flat cloud becomes a tight curve when the right non-dimensional variables are used.

End-to-End Example: Drag on a Sphere

We expect the steady drag force F_D on a sphere to depend on fluid density ρ, dynamic viscosity η, characteristic length L (diameter), and speed v. List dimensions

$$[F_D] = \mathrm{M\,L\,T^{-2}}, \quad [\rho] = \mathrm{M\,L^{-3}}, \quad [\eta] = \mathrm{M\,L^{-1}\,T^{-1}}, \quad [L] = \mathrm{L}, \quad [v] = \mathrm{L\,T^{-1}}.$$

There are $n = 5$ variables and $k = 3$ base dimensions, so expect $n - k = 2$ independent π groups. Choose repeating variables (ρ, v, L) (they span $\mathrm{M, L, T}$). Build

$$\pi_1 = \frac{F_D}{\rho v^2 L^2} \quad \text{(dimensionless drag coefficient } C_D\text{)}, \qquad \pi_2 = \frac{\rho v L}{\eta} \quad \text{(Reynolds number Re)}.$$

The Buckingham π–theorem says $\pi_1 = f(\pi_2)$, i.e.

$$\boxed{C_D = f(\mathrm{Re})}$$

which is the classic empirical relationship: in creeping flow (small Re), $C_D \sim 24/\mathrm{Re}$ (Stokes); at larger Re, C_D flattens to an $\mathcal{O}(1)$ constant. The moral: the dimensionless plot C_D vs. Re collapses experiments of different sizes and speeds.

15.3 The Pendulum Period (Dimensional Guess)

At small angles, a simple pendulum's period T depends primarily on gravity g and length L; mass and small-amplitude do not matter. With $[T] = \mathrm{T}$, $[g] = \mathrm{L\,T^{-2}}$, and $[L] = \mathrm{L}$, the only combination with time units is

$$T \sim \sqrt{\frac{L}{g}}.$$

The exact solution supplies the constant 2π, but the *shape* $T \propto L^{1/2}$ falls straight out of dimensions. The message is practical: long pendulums swing slowly; short ones swing quickly. If a prediction violates this by suggesting $T \propto L$, dimensions flag a mistake before you compute.

As shown in Figure 15.1, a log–log plot of T versus L has slope $1/2$.

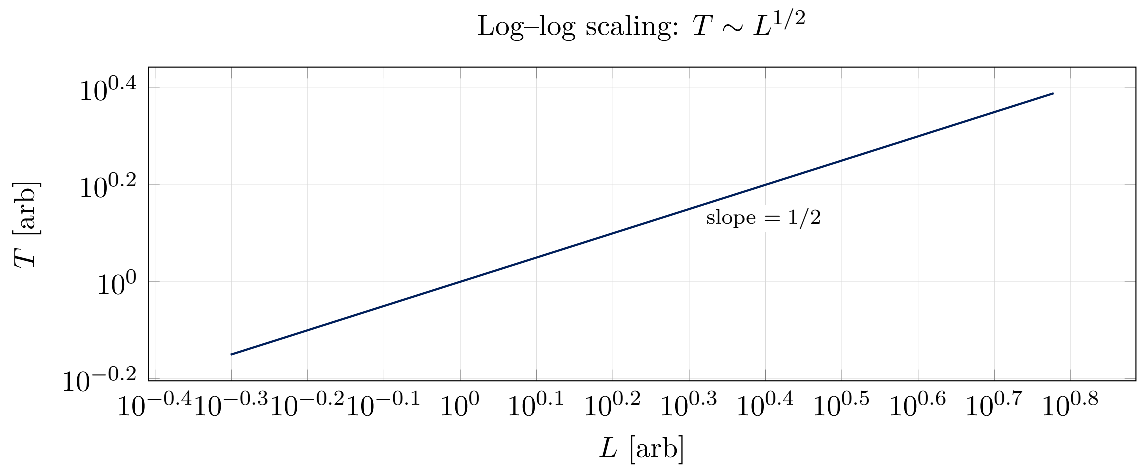


Figure 15.1: On log–log axes, power laws become straight lines; the slope is the exponent.

15.4 Quick π–Theorem Workflow

When many variables appear, the Buckingham π–theorem says you can rewrite a relationship among n variables with k base dimensions as a relation among $n-k$ independent dimensionless combinations (π groups). A practical 4-step recipe:

1. List variables and their dimensions.
2. Choose k repeating variables spanning the base dimensions.
3. Build $n-k$ independent π groups (products that come out dimensionless).
4. Write a relation among the π's; test/fit with data or limiting cases.

Lead-in example: drag coefficient C_D is already dimensionless; Reynolds number $\mathrm{Re} = \dfrac{\rho v L}{\eta}$ distinguishes flow regimes.

Picking Repeating Variables

Choose k repeating variables that span the base dimensions (e.g., M, L, T) and are natural controls rather than outcomes. In many fluid problems (ρ, v, L) serve well, yielding $\mathrm{Re} = \frac{\rho v L}{\eta}$. For gravity-wave or ship-model studies, include g and build the Froude number $\mathrm{Fr} = \frac{v}{\sqrt{gL}}$ alongside Re.

15.5 Scaling Intuition

If length scales by λ, areas scale by λ^2, volumes by λ^3. Weight scales like λ^3 but cross-sectional strength like λ^2—one reason small creatures seem "stronger" for their size. In fluids, $\mathrm{Re} \propto L$ at fixed v, ρ, η—bigger systems more easily become turbulent. On free surfaces (waves, ships), the gravity–inertia balance is captured by Froude $\mathrm{Fr} = \frac{v}{\sqrt{gL}}$; model tests match Fr to preserve wave patterns even if Re cannot be matched simultaneously (see Chapter 14).

Worked Example: Matching a Wind-Tunnel Model

A full car of length $L_{\text{full}} = 4.5\,\text{m}$ cruising at $v_{\text{road}} = 30\,\text{m/s}$ in air ($\rho \approx 1.2\,\text{kg/m}^3$, $\mu \approx 1.8 \times 10^{-5}\,\text{Pa s}$) runs at $\mathrm{Re}_{\text{full}} = \rho v_{\text{road}} L_{\text{full}}/\mu \approx 9.0 \times 10^6$. A 1:5 scale model has $L_{\text{model}} = 0.90\,\text{m}$. To match Re you must increase the tunnel speed to

$$v_{\text{model}} = \frac{\mathrm{Re}_{\text{full}}\,\mu}{\rho\,L_{\text{model}}} \approx 150\,\text{m/s},$$

roughly Mach 0.45. If the tunnel is limited to $60\,\text{m/s}$, the achievable Reynolds number is $\mathrm{Re}_{\text{model}} \approx 3.6 \times 10^6$, four times smaller. Measurements still trend correctly, but you must apply a scaling correction (or use a denser/colder fluid) before quoting road data. Figure 15.2 sketches the geometry and the Reynolds-number bookkeeping.

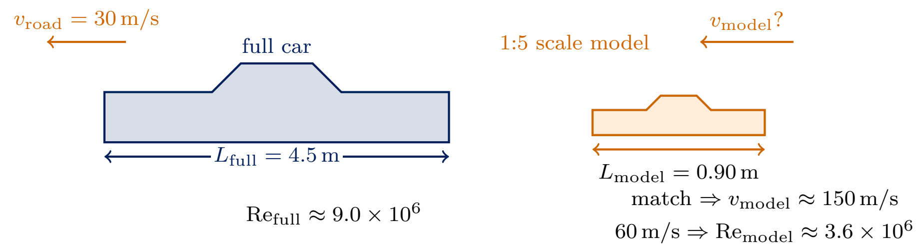


Figure 15.2: Scaling a 1:5 model: matching Reynolds numbers demands $v_{\text{model}} \approx 150\,\text{m/s}$. Lower tunnel speeds mean lower Re and require corrections.

15.6 Exercises

Short problems to practice and check:

1. **Check it.** Verify that kinetic energy $K = \frac{1}{2}mv^2$ has units of work.
2. **Pendulum.** Use dimensions to guess how the period depends on L and g.
3. **Drag guess.** For quadratic drag, argue that $v_t \sim \sqrt{mg/(\rho A)}$.

15.7 Summary and Review

A quick checklist before moving on:

- Dimensional consistency is a must-have filter for equations.
- π groups reduce variables to dimensionless combinations.
- Scaling clarifies how size changes behavior; power laws show straight lines on log–log plots.

15.8 Where We're Heading Next

In Chapter 16 we add simple numerical methods and modelling workflows to turn equations into runnable predictions, with practical checks for accuracy, stability, and energy behavior.

Common Pitfalls

Confusing units with dimensions; mixing measurement systems mid-problem; forgetting to report dimensionless combinations; choosing repeating variables that do not span base dimensions.

Try in 60 seconds

Quick checks:

- List the dimensions of g, ρ, and dynamic viscosity η.
- If length doubles, how do mass and surface area scale? Explain.

Chapter 16

Numerical Methods for Newtonian Systems

Analytic solutions are wonderful—but many problems need approximate time-stepping. We discretize time, update positions and velocities, and watch accuracy/stability as we go.

Learning Objectives

You will discretize an ODE in time, implement simple updates (explicit Euler, semi-implicit/symplectic Euler), and read stability/accuracy effects from step size.

Symbols at a Glance

Δt time step, $t_n = n\,\Delta t$, $x_n \approx x(t_n)$, $v_n \approx v(t_n)$.

Analogy: Frame-by-Frame Animation

Instead of a continuous movie, draw the next frame from the current one using a simple rule. Smaller frame jumps (Δt) make smoother motion.

16.1 Discretizing Time

For $\dot{x} = f(x,t)$, explicit Euler is $x_{n+1} = x_n + f(x_n, t_n)\,\Delta t$. For mechanics, the first-order system

$$\dot{x} = v, \qquad \dot{v} = a(x, v, t)$$

updates as

$$v_{n+1} = v_n + a(x_n, v_n, t_n)\,\Delta t, \qquad x_{n+1} = x_n + v_n\,\Delta t.$$

The semi-implicit/symplectic variant uses the new velocity in the position update: $x_{n+1} = x_n + v_{n+1}\,\Delta t$—often more stable for oscillations.

Terminology: the *local truncation error* (error made in one step assuming perfect input) is $\mathcal{O}(\Delta t^2)$ for Euler; accumulated *global error* over an interval is $\mathcal{O}(\Delta t)$. Halving Δt should roughly halve the global error for first-order methods (see Appendix Appendix E).

Before Figure 16.1, recall the schematic from Chapter 2: staircase vs. smooth.

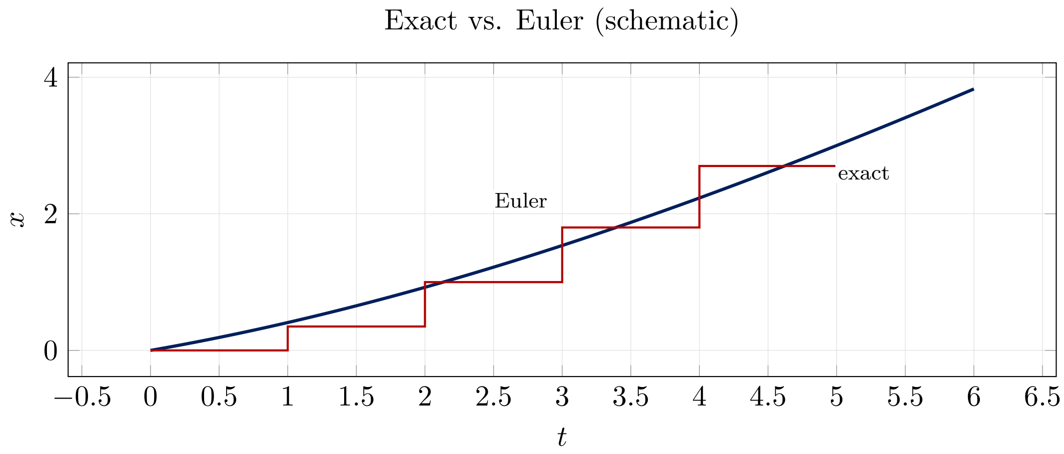


Figure 16.1: Schematic comparison: exact smooth curve vs. explicit Euler steps.

16.2 Accuracy and Stability

Smaller Δt typically improves accuracy (local error $\sim \Delta t^2$ for Euler; global $\sim \Delta t$). For oscillatory or stiff problems, naive Euler can blow up; symplectic Euler often behaves qualitatively better for conservative systems.

Linear stability lens. For the test equation $\dot{y} = \lambda y$ with $\Re(\lambda) < 0$, explicit Euler gives $y_{n+1} = (1+\lambda\Delta t)y_n$. This decays only if $|1+\lambda\Delta t| < 1$, which bounds Δt. For oscillations (purely imaginary λ), $|1+i\omega\Delta t| > 1$ so Euler grows—explaining energy drift. Structure-preserving (symplectic) schemes keep the phase-space area and tend to bound energy oscillations in Hamiltonian systems.

A Concrete Example: Simple Harmonic Oscillator

Consider $\ddot{x} + \omega^2 x = 0$ with $\omega = 1$, $x(0) = 1$, $v(0) = 0$ so the exact solution is $x(t) = \cos t$. The total energy $E = \frac{1}{2}(v^2 + x^2)$ should stay constant.

Before Figure 16.2, keep in mind what "good" looks like: a nearly horizontal energy trace. A steadily rising or falling trace means the method is injecting or removing energy.

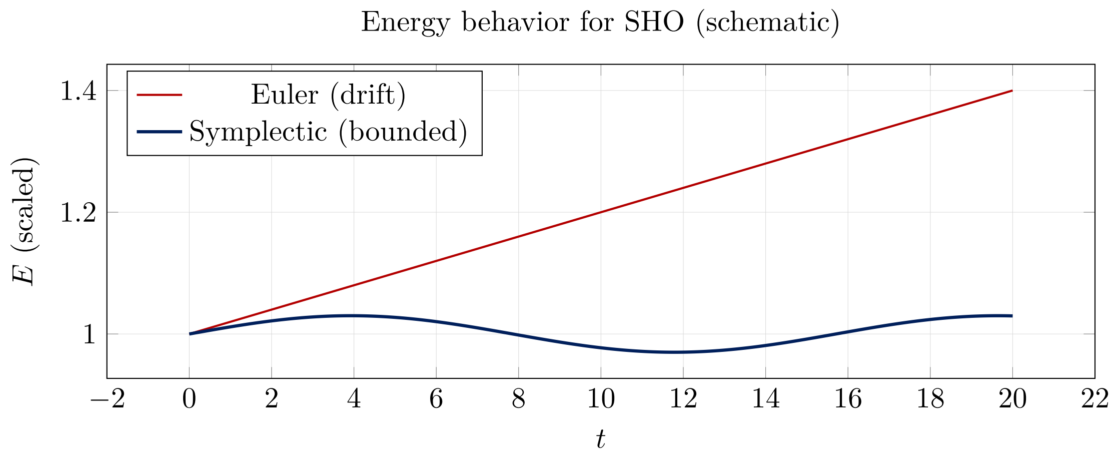


Figure 16.2: For the SHO, explicit Euler tends to drift in energy, while symplectic Euler nearly conserves it (oscillates around the true constant).

Worked Example: Symplectic Euler Tames a Pendulum

A 1 m pendulum released from 15° obeys $\ddot{\theta} + \theta = 0$ (small-angle). Step it for 10 s with $\Delta t = 0.1$ s. Explicit Euler updates $\theta_{n+1} = \theta_n + \omega_n \Delta t$, $\omega_{n+1} = \omega_n - \theta_n \Delta t$; symplectic Euler uses $\omega_{n+1} = \omega_n - \theta_n \Delta t$, then $\theta_{n+1} = \theta_n + \omega_{n+1} \Delta t$. Both are first order, yet explicit Euler's energy triples while symplectic Euler keeps it near the initial $E_0 \approx 3.4 \times 10^{-2}$. See Figure 16.3 for the angle history and energy comparison.

16.3 Practical Workflow and Checks

Simple habits raise confidence:

- Start coarse, then halve Δt and compare: stable results should converge.
- Track conserved or monotone quantities (e.g., energy in conservative systems) as qualitative checks.
- Beware floating-point roundoff with very small Δt or huge step counts; compare against an analytic or high-accuracy reference when possible.

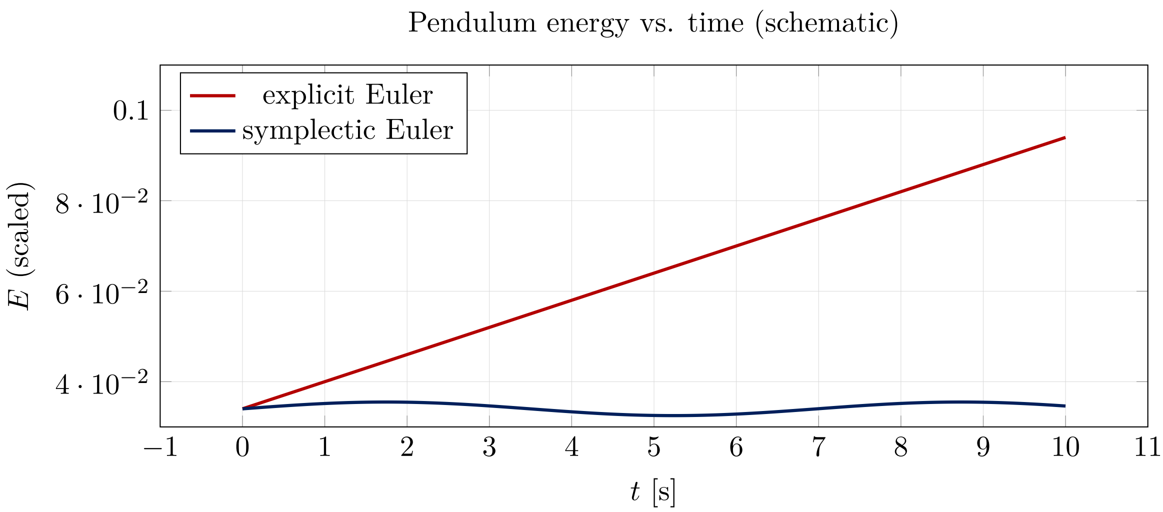


Figure 16.3: For a small-angle pendulum, explicit Euler tends to inject energy (rising trace), while symplectic Euler keeps energy nearly constant (small bounded oscillation).

Error shrinks as Δt shrinks. On log–log axes, first-order global error curves are straight lines with slope one, as shown in Figure 16.4.

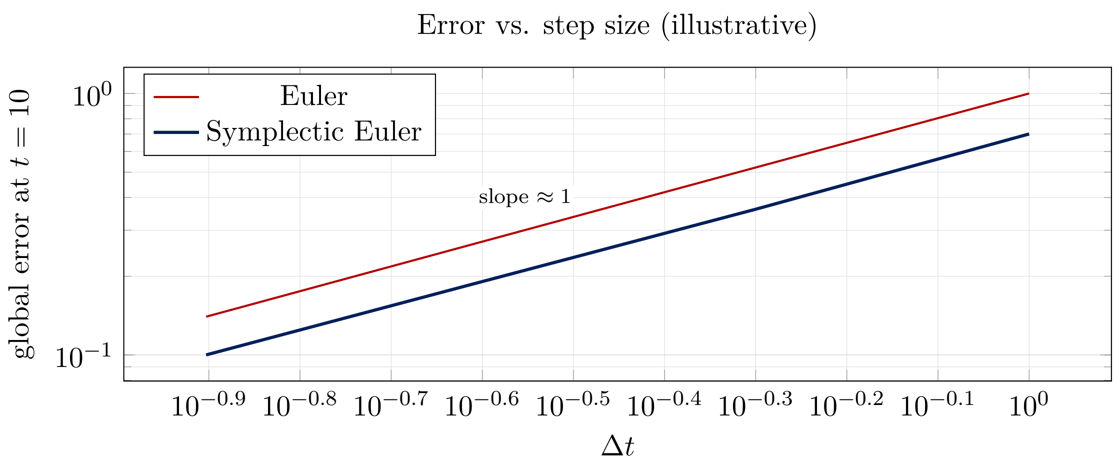


Figure 16.4: Both explicit and symplectic Euler have global error that scales $\propto \Delta t$; the symplectic variant often enjoys a smaller constant for oscillatory problems.

16.4 Worked Python Examples

Short, self-contained scripts illustrate how the updates look in code. You can copy–paste them to a file and run with any Python 3.x. We annotate each step so the mapping to the math is clear.

Simple Harmonic Oscillator: Euler vs. Symplectic Euler

We integrate $\ddot{x} + x = 0$ for $t \in [0, 20]$ with $\Delta t = 0.05$ and compare energies.

Python: SHO — Euler vs. Symplectic

```python
# Simple harmonic oscillator: x'' + x = 0
# Compare explicit Euler and symplectic Euler on energy behavior

import math

def euler_sho(x0=1.0, v0=0.0, dt=0.05, t_end=20.0):
    x, v = x0, v0
    xs, vs, Es = [], [], []
    t = 0.0
    while t <= t_end + 1e-12:
        # Energy E = 0.5*(v^2 + x^2) should be constant for exact solution
        Es.append(0.5*(v*v + x*x))
        xs.append(x); vs.append(v)
        # Explicit Euler updates use acceleration a = -x evaluated at (x_n, v_n)
        a = -x
        v = v + a*dt      # v_{n+1} = v_n + a_n dt
        x = x + v*0.0 + (v - a*dt)*dt  # equivalent to x_{n+1} = x_n + v_n dt
        # (written this way to highlight it's the old v; compact form below)
        # x = x + vs[-1]*dt
        t += dt
    return xs, vs, Es

def symplectic_euler_sho(x0=1.0, v0=0.0, dt=0.05, t_end=20.0):
    x, v = x0, v0
    xs, vs, Es = [], [], []
    t = 0.0
    while t <= t_end + 1e-12:
        Es.append(0.5*(v*v + x*x))
        xs.append(x); vs.append(v)
```

```
        # Symplectic Euler uses new v to update x; still a = -x at the start
        a = -x
        v = v + a*dt          # v_{n+1} = v_n + a_n dt
        x = x + v*dt          # x_{n+1} = x_n + v_{n+1} dt
        t += dt
    return xs, vs, Es

if __name__ == "__main__":
    # Run both methods and print final energies for a quick comparison
    _, _, E_eu = euler_sho()
    _, _, E_sy = symplectic_euler_sho()
    print(f"Euler   final E: {E_eu[-1]:.6f}  (start {E_eu[0]:.6f})")
    print(f"Symplec final E: {E_sy[-1]:.6f}  (start {E_sy[0]:.6f})")
```

You should observe Euler's energy drifting noticeably while the symplectic method oscillates around the initial value.

Projectile with Linear Drag (Component Form)

We model $m\dot{\vec{v}} = \vec{F} = \langle 0, -mg\rangle - c\,\vec{v}$ with $m = 1$, $g = 9.81$, $c = 0.2$. This shows how to structure vector updates.

Python: 2D Projectile with Linear Drag

```
# 2D projectile under linear drag: v' = g - (c/m) v
# Semi-implicit (symplectic) Euler on velocity; then update position

import math

def projectile_drag(x0=(0.0, 0.0), v0=(20.0, 16.0), g=9.81, c=0.2, m=1.0,
                    dt=0.02, t_end=3.0):
    x, y = x0
    vx, vy = v0
    traj = []  # store (t, x, y, vx, vy)
    t = 0.0
    while t <= t_end + 1e-12:
        traj.append((t, x, y, vx, vy))
        # Acceleration components from gravity and linear drag
        ax = -(c/m)*vx
        ay = -g - (c/m)*vy
        # Update velocities first (symplectic-like)
        vx = vx + ax*dt
        vy = vy + ay*dt
        # Then update positions using the new velocities
        x = x + vx*dt
        y = y + vy*dt
        t += dt
        # Stop if projectile hits the ground (simple termination)
        if y < 0.0:
            break
    return traj

if __name__ == "__main__":
    data = projectile_drag()
    # Show a few samples to verify the trajectory trends
    for k in range(0, len(data), max(1, len(data)//5)):
        t, x, y, vx, vy = data[k]
        print(f"t={t:4.2f}  x={x:6.2f}  y={y:6.2f}  vx={vx:6.2f}  vy={vy:6.2f}")
```

Both examples follow the same state-update pattern used throughout the book; only the acceleration function changes with the physics.

16.5 Exercises

Practice small steps and comparisons.

1. **Euler step.** For $\ddot{x} = -\omega^2 x$ with $\omega = 1$, $x_0 = 1$, $v_0 = 0$, $\Delta t = 0.1$, compute (x_1, v_1) by explicit Euler and by symplectic Euler.
2. **Step effect.** Halve Δt and compare $x(1)$ against the exact $\cos 1$.

16.6 Summary and Review

A quick checklist before moving on:

- Explicit Euler: easy but only first-order accurate and can be unstable.
- Symplectic Euler: similar cost, better qualitative behavior for Hamiltonian systems.
- Smaller Δt improves accuracy but increases cost.

16.7 Where We're Heading Next

These numerical tools support many chapters ahead and pair naturally with the math appendices. For a compact reference of methods and stability/error heuristics, see Appendix E.

Common Pitfalls

Mismatching x and v updates; forgetting to evaluate accelerations at the correct state; confusing local with global error; using too large Δt.

Try in 60 seconds

Quick drills:

- Write the symplectic Euler update for x and v explicitly.
- For $\ddot{x} = -x$, estimate $x(0.2)$ with $\Delta t = 0.1$.

Epilogue — What Comes After Newtonian Physics?

Newtonian mechanics is astonishingly successful. On everyday scales—with speeds much less than the speed of light, moderate gravitational fields, and objects large enough to ignore quantum granularity—it predicts motion with crisp accuracy and remarkable economy. Yet nature stretches beyond these conditions. Here is a compact map of where Newtonian ideas bend and how modern physics extends them.

Where It Breaks

- **Very fast** ($v \sim c$). At speeds approaching the speed of light, time dilates and lengths contract. Newton's velocity addition and absolute time fail.
- **Very strong gravity / very large scales.** Near massive bodies or across cosmological distances, space and time curve. Newton's instantaneous action at a distance is replaced by geometry.
- **Very small** (atomic and subatomic). Energy comes in quanta, particles behave like waves, and measurement itself carries probabilistic structure.

Three Great Extensions

- **Special Relativity** (Einstein, 1905). Mechanics and electromagnetism share the same speed limit c. Space and time form spacetime; energy and momentum combine into four-vectors. Newton's second law survives in relativistic dress, and Newtonian results reappear when $v \ll c$.
- **General Relativity** (Einstein, 1915). Gravity is curved spacetime. Free fall is motion along geodesics. Newton's law of gravitation emerges as a weak-field, low-speed approximation.
- **Quantum Mechanics** (1920s–). States are waves in Hilbert space; observables are operators; outcomes are probabilistic with strict rules. Classical mechanics returns in the limit of large quantum numbers or small $\hbar$ (the correspondence principle).

How the Pieces Fit

In practice, physics is a set of overlapping maps, each valid on its domain with clean bridges between them:

- *Classical (Newtonian)* for everyday speeds, weak gravity, and macroscopic objects.
- *Relativistic mechanics* for fast motion or precise timing (GPS satellites, particle beams).

- *Quantum mechanics* for atoms, molecules, and materials; *quantum field theory* for particle physics.
- *Classical field theories* (fluids, elasticity, electromagnetism) describe collective behavior where particle details are hidden.

Each newer theory contains Newtonian mechanics as a limiting case. Learning to recognize which map applies—and when to change maps—is a professional superpower.

What We Still Don't Have

Despite a century of progress there is no single, complete *theory of everything* (TOE). The Standard Model of particle physics and general relativity coexist but resist unification. Dark matter and dark energy signal phenomena beyond what current theories explain. This is not a failure; it is an invitation.

Takeaway

Newton's framework remains the right starting point for most engineered and natural systems you will encounter. It is also the common language you'll use when stepping into relativity, quantum theory, or continuum descriptions. Master the Newtonian map; then explore the others with confidence, knowing where each begins and where the next takes over.

Part VIII
Mathematics for Newtonian Mechanics

Appendix Overview

A concise toolkit for results used throughout the book.

- **Calculus for motion** (limits, derivatives, integrals, Taylor tools) complements kinematics and dynamics in Chapters 3, 4 and 8.
- **Vectors and linear algebra** (components, dot/cross, matrices) support multi-D motion and forces in Chapters 5, 7 and 11.
- **Multivariable basics** (grad, div, curl; line/surface integrals) tie into fields and potentials in Chapter 12.
- **ODE essentials** (existence, stability, linear systems) back up modelling across Chapters 13 and 16.
- **Numerics quick reference** (floating-point, step-size, error) complements the methods in Chapter 16.

Each section is a quick reference with minimal proofs and pointers back to where the ideas are first used in the main text.

Appendix A

Calculus Essentials for Motion

This appendix gathers exactly the one–variable calculus tools used throughout the book, with pictures first and formulas second. If you want a memory hook: slope means rate; area means accumulation. When in doubt, re-draw the picture.

Learning Objectives

You can read limits and continuity from a graph, compute derivatives with core rules, use small Taylor expansions, evaluate definite integrals, and apply the Fundamental Theorem of Calculus (FTC) to turn rates into totals.

Symbols at a Glance

t time; $x(t)$ position; $v(t) = \dot{x}$ velocity; $a(t) = \dot{v}$ acceleration; $\mathrm{d}x$ differential; $\int f\,\mathrm{d}t$ integral.

Analogy: Slope and Paint

Slope is how steep the road is at your feet; area is how much paint it takes to cover a fence of height $f(t)$ as you walk from $t = a$ to $t = b$.

A.1 Differentiation

We start with precise definitions and then build geometric intuition.

Definition 1 (First Derivative)**.** Let f be defined on an interval containing t_0. We say f is differentiable at t_0 with derivative $f'(t_0)$ if the limit

$$f'(t_0) = \lim_{h\to 0} \frac{f(t_0 + h) - f(t_0)}{h}$$

exists. Intuitively, $f'(t_0)$ is the slope of the tangent line to the graph of f at t_0; in mechanics, it is a rate (e.g., velocity).

Definition 2 (Second Derivative)**.** If f' exists in a neighborhood of t_0 and is differentiable at t_0, the second derivative is

$$f''(t_0) = \lim_{h\to 0} \frac{f'(t_0 + h) - f'(t_0)}{h}.$$

Geometrically, $f''(t_0)$ measures curvature (how the slope changes). In kinematics, $x''(t) = a(t)$ is acceleration (Chapter 3).

Before Figure A.1, remember: f' is slope and f'' is curvature.

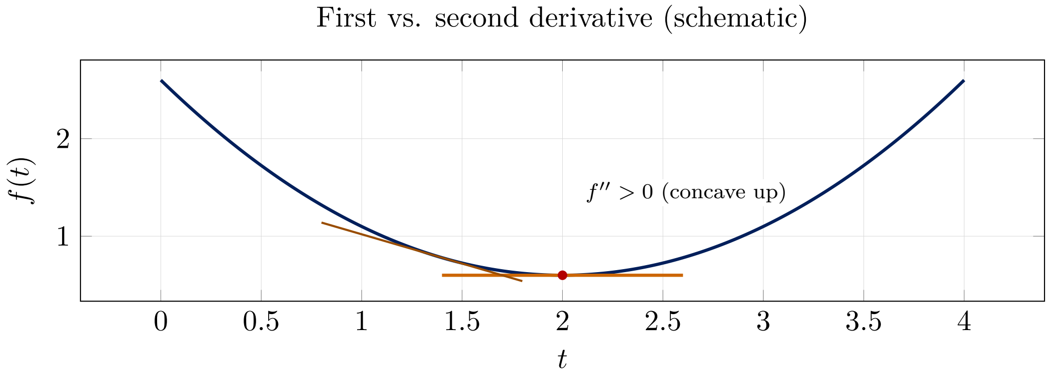


Figure A.1: Schematic: f' is the slope of the tangent; f'' describes how the slope changes (curvature).

A.2 Derivative Rules You Actually Use

We differentiate to turn position into velocity (Chapter 2). The rules you reach for most often are the ones that let you simplify expressions quickly without losing sight of the physical meaning (rate of change in time):

Lead-in: here are the core rules, with examples you can apply immediately.

- Linearity: $(af + bg)' = af' + bg'$. Example: $\frac{\mathrm{d}}{\mathrm{d}t}(2t + 3\sin t) = 2 + 3\cos t$.
- Product: $(fg)' = f'g + fg'$. Example: $\frac{\mathrm{d}}{\mathrm{d}t}(t\sin t) = \sin t + t\cos t$.
- Chain: $(f \circ g)' = (f' \circ g)\, g'$. Example: $\frac{\mathrm{d}}{\mathrm{d}t}\sin(t^2) = \cos(t^2) \cdot 2t$.

Small-angle reminders we actually use (Chapters 6 and 13):

- For $|\theta| \ll 1$ (*radians*): $\sin\theta \approx \theta$, $\cos\theta \approx 1 - \frac{\theta^2}{2}$. Always use radians for calculus with trig.

A.3 Limits and Continuity

Intuitively, $f(t)$ approaches L as $t \to t_0$ if we can make $f(t)$ as close to L as we like by taking t sufficiently close to t_0. Continuity at t_0 means the graph has no jump or hole there: $\lim_{t\to t_0} f(t) = f(t_0)$.

In practice, we rely on two habits:

- Look left and right: do one-sided limits agree? If yes and the point value matches, the function is continuous.
- Replace scary algebra with a picture: zoom into the point on the graph. If the zoom looks like a straight line, a derivative likely exists.

Before Figure A.2, recall: the derivative is the *limit* of secant slopes.

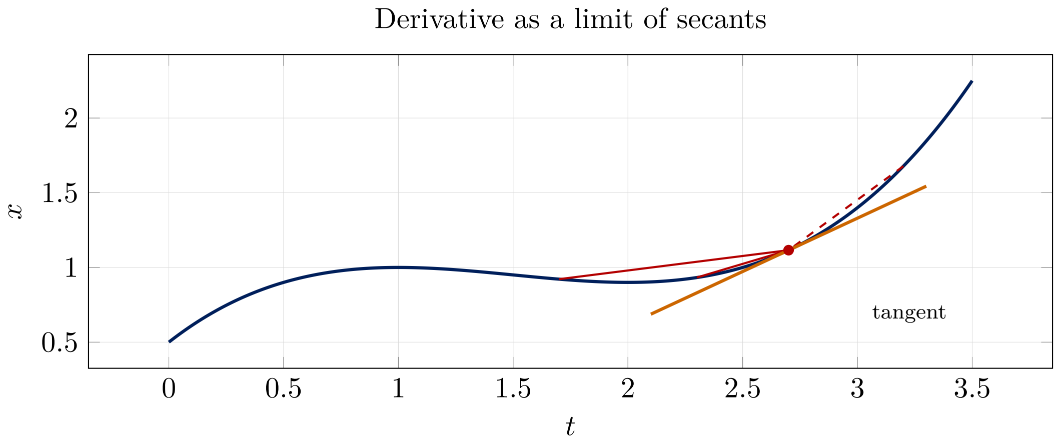


Figure A.2: Secants converge to the tangent at a point. Bold secants from the left. Dashed secant from the right.

A.4 Taylor Approximations (Local Models)

Near a point t_0, $f(t)$ is well-approximated by a low-order polynomial built from derivatives at t_0. Think "best straight line," then "best gentle bend." For smooth f,

$$f(t) \approx f(t_0) + f'(t_0)(t - t_0) + \tfrac{1}{2}f''(t_0)(t - t_0)^2.$$

Engineers read this as "straight line + gentle bend." In Chapter 13, small-angle pendulum motion is nothing but a Taylor approximation to the sine.

Before Figure A.3, remember: approximations are *local*. Step too far, and the picture bends away.

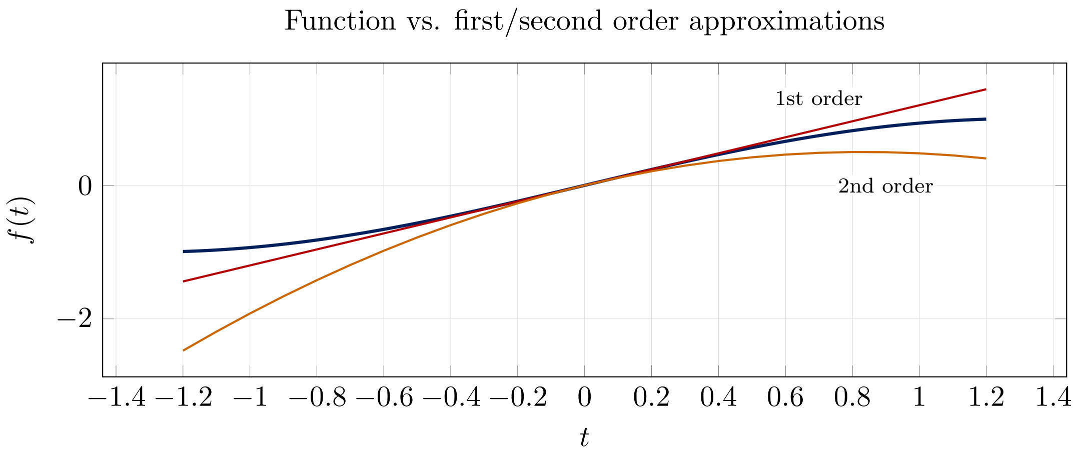


Figure A.3: Local linear and quadratic models of a smooth function; good near $t_0 = 0$, worse as you move away.

A.5 Integration

First a general definition, then a picture.

Definition 3 (Riemann–Stieltjes Integral)**.** Let f be bounded on $[a, b]$ and α be an increasing function on $[a, b]$. For a partition $\mathcal{P}\colon a = t_0 < \cdots < t_n = b$ and sample points $\xi_i \in [t_{i-1}, t_i]$, form sums

$$S(\mathcal{P}, \xi; f, \alpha) = \sum_{i=1}^{n} f(\xi_i)\,\big(\alpha(t_i) - \alpha(t_{i-1})\big).$$

If these sums converge to a common limit as the mesh $\|\mathcal{P}\| \to 0$ (independent of choices of ξ_i), we call the limit the Riemann–Stieltjes integral and write

$$\int_a^b f\,\mathrm{d}\alpha.$$

When $\alpha(t) = t$, this reduces to the usual Riemann integral $\int_a^b f(t)\,\mathrm{d}t$.

Before Figure A.4, keep in mind: the Riemann–Stieltjes sum $\sum f(\xi_i)\,\Delta\alpha_i$ weights samples by increments of an increasing integrator α. Equal steps in α generally mean uneven steps in t.

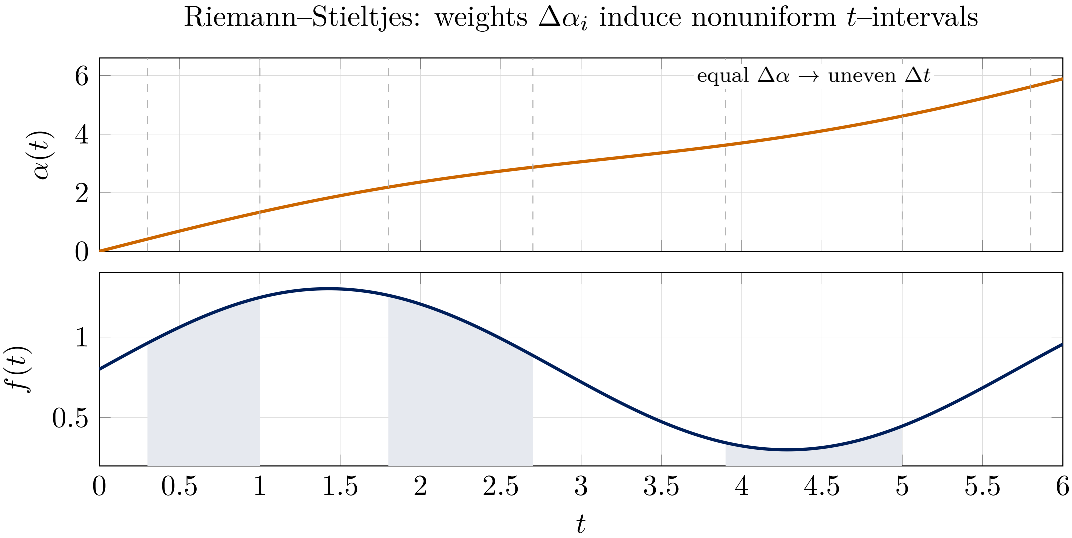


Figure A.4: Schematic: equal weights in α select nonuniform time intervals. Summing f times $\Delta\alpha$ generalizes "area under a curve."

A.6 Fundamental Theorems (How They Fit)

The two operations *differentiation* and *integration* are inverses under suitable assumptions. We collect the relationships used throughout the book and visualize the area–as–accumulation idea.

- Fundamental Theorem of Calculus (part I). If $F'(t) = f(t)$ and f is integrable,

$$\int_a^b f(t)\,\mathrm{d}t = F(b) - F(a).$$

- Fundamental Theorem of Calculus (part II). If f is integrable and sufficiently nice,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_a^t f(\tau)\,\mathrm{d}\tau = f(t).$$

- Riemann–Stieltjes with differentiable integrator. If α is differentiable with $\alpha' \in L^1$, then

$$\int_a^b f\,\mathrm{d}\alpha = \int_a^b f(t)\,\alpha'(t)\,\mathrm{d}t,$$

and the Leibniz rule extends to

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_a^t f(\tau)\,\mathrm{d}\alpha(\tau) = f(t)\,\alpha'(t)$$

when the hypotheses hold (see also Chapter 8).

- Integration by parts mirrors the product rule: for suitable f, g,

$$\int_a^b f\,\mathrm{d}g = f(b)g(b) - f(a)g(a) - \int_a^b g\,\mathrm{d}f.$$

Before Figure A.5, keep the picture in mind: the integral is the *signed* area. As introduced in Chapter 3, area under $a(t)$ gives the change in v, and area under $v(t)$ gives the change in x.

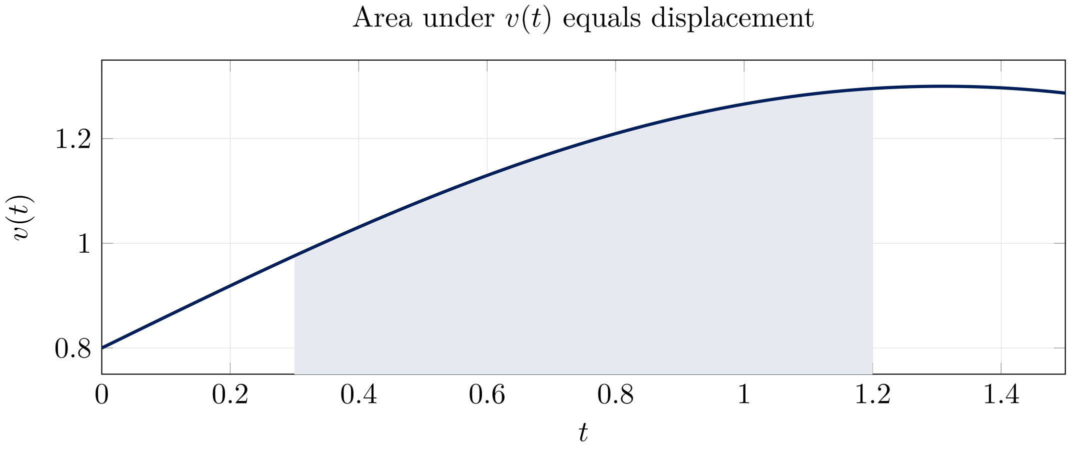


Figure A.5: Shaded area under $v(t)$ between two times equals displacement.

A.7 Integrals and the FTC (Mechanics Lens)

The Fundamental Theorem of Calculus (FTC) ties rates to totals. If $F'(t) = f(t)$ and f is integrable on $[a, b]$, then

$$\int_a^b f(t)\,\mathrm{d}t = F(b) - F(a).$$

In mechanics: $v = \dot{x}$ so $x(b) - x(a) = \int_a^b v(t)\,\mathrm{d}t$ (displacement is area under v); $a = \dot{v}$ so $v(b) - v(a) = \int_a^b a(t)\,\mathrm{d}t$. Piecewise-smooth signals are fine: add the areas across pieces.

A.8 Worked Examples

The following presents worked examples to illustrate the relationship between differentiation and integration. First, application of integration to a velocity curve.

Worked Example: From $v(t)$ to $x(t)$

We know from the Fundamental Theorem of Calculus that displacement is the area under the velocity curve. Suppose

$$v(t) = 2 + 0.8\,t \quad \text{for } 0 \le t \le 3, \qquad x(0) = 1.$$

Then the displacement over $[0, 3]$ is

$$\int_0^3 v(t)\,\mathrm{d}t = \int_0^3 \big(2 + 0.8\,t\big)\,\mathrm{d}t = \Big[2t + 0.4\,t^2\Big]_0^3 = 6 + 3.6 = 9.6.$$

Hence the new position is $x(3) = x(0) + 9.6 = 10.6$. If instead we only knew acceleration $a(t)$, we could integrate twice: first get v from a with $v(0)$, then get x from v with $x(0)$.

Second, application of differentiation to a position curve.

Worked Example: From $x(t)$ to $v(t)$

Differentiation turns position into velocity. Suppose

$$x(t) = 1 + 2t + 0.4\,t^2 \quad \text{for } t \ge 0.$$

Then by definition $v(t) = \dfrac{\mathrm{d}x}{\mathrm{d}t} = 2 + 0.8\,t$. If you want the acceleration as well, differentiate once more to get

$$a(t) = \frac{\mathrm{d}v}{\mathrm{d}t} = 0.8 \text{ (a constant)}.$$

As a quick check with the previous example, integrating this $v(t)$ from 0 to 3 reproduces the same displacement $\Delta x = 9.6$.

A.9 Techniques We Actually Use

Lead-in: a tiny toolkit goes a long way in mechanics.

- Substitution: straighten a composition. Example: $\int 2t\cos(t^2)\,\mathrm{d}t = \int \cos u\,\mathrm{d}u = \sin u + C$ with $u = t^2$.
- Integration by parts: trade derivative for antiderivative. Example: $\int te^t\,\mathrm{d}t = te^t - \int e^t\,\mathrm{d}t = e^t(t-1) + C$.

Common Pitfalls

Confusing local linearity with global behavior; forgetting that Taylor expansions are local; mixing up displacement (signed area) with distance (always non-negative); dropping the chain rule in composites; integrating with wrong limits.

Try in 60 seconds

Quick checks:

- **Slope vs. area.** Which graph quantity gives Δx from $a(t)$? From $v(t)$?
- **Small angle.** Use $\sin\theta \approx \theta$ to linearize a pendulum near rest (see Chapter 13).
- **FTC check.** If $F'(t) = f(t)$, what is $\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t f(\tau)\,\mathrm{d}\tau$?

Appendix B

Vectors and Linear Algebra Essentials

This appendix gives only the vector tools we actually use in the main text. Pictures lead; formulas follow. Keep three verbs in mind: add (combine), project (compare), rotate (re-express).

Learning Objectives

You can add and scale vectors, compute dot and cross products with geometric meaning, project one vector onto another, read simple rotation matrices in 2D, and connect torque and area to cross products.

Symbols at a Glance

$\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{r}, \boldsymbol{v}$ vectors (bold); $\|\boldsymbol{a}\|$ magnitude; $\hat{\boldsymbol{u}}$ unit vector; $\boldsymbol{a} \cdot \boldsymbol{b}$ dot product; $\boldsymbol{a} \times \boldsymbol{b}$ cross product (3D); $R(\theta)$ 2D rotation matrix.

Analogy: Shadows and Spins

The dot product is a *shadow*—how much one arrow lies along another. The cross product is a *spin cue*—how big the parallelogram is and which way a screw would turn.

B.1 Vectors and Basic Operations

A vector in the plane is an ordered pair $\boldsymbol{a} = (a_x, a_y)$; in space, $\boldsymbol{a} = (a_x, a_y, a_z)$. Add components to add vectors; scale components to scale a vector. In 2D, $\|\boldsymbol{a}\| = \sqrt{a_x^2 + a_y^2}$; in 3D, $\|\boldsymbol{a}\| = \sqrt{a_x^2 + a_y^2 + a_z^2}$. The unit vector is $\hat{\boldsymbol{a}} = \boldsymbol{a}/\|\boldsymbol{a}\|$ when $\boldsymbol{a} \neq \boldsymbol{0}$.

Before Figure B.1, recall: tip-to-tail addition is geometry you can see.

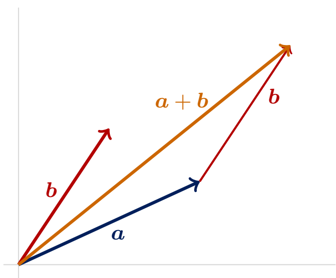


Figure B.1: Tip-to-tail addition: place $\boldsymbol{b}$ at the tip of $\boldsymbol{a}$; the diagonal is $\boldsymbol{a} + \boldsymbol{b}$.

B.2 Dot Product and Projection

The dot product measures alignment: $\boldsymbol{a}\cdot\boldsymbol{b} = \|\boldsymbol{a}\|\,\|\boldsymbol{b}\|\cos\phi$, where ϕ is the angle between them. Algebraically in 2D/3D, $\boldsymbol{a}\cdot\boldsymbol{b} = a_x b_x + a_y b_y (+a_z b_z)$. The projection of $\boldsymbol{a}$ onto $\hat{\boldsymbol{u}}$ is $(\boldsymbol{a}\cdot\hat{\boldsymbol{u}})\,\hat{\boldsymbol{u}}$.

We use this for work in Chapter 8: $W = \int \boldsymbol{F}\cdot d\boldsymbol{r}$—force along displacement.

Before Figure B.2, keep in mind: the dot is the signed length of the shadow of one vector on another.

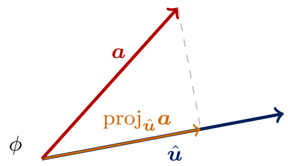


Figure B.2: Projection: $\boldsymbol{a}\cdot\hat{\boldsymbol{u}} = \|\boldsymbol{a}\|\cos\phi$ is the signed length of the shadow of $\boldsymbol{a}$ along $\hat{\boldsymbol{u}}$.

B.3 Cross Product, Area, and Torque

In 3D, $\boldsymbol{a}\times\boldsymbol{b}$ is perpendicular to both $\boldsymbol{a}$ and $\boldsymbol{b}$, with magnitude $\|\boldsymbol{a}\times\boldsymbol{b}\| = \|\boldsymbol{a}\|\,\|\boldsymbol{b}\|\sin\phi$, equal to the area of the parallelogram spanned by $\boldsymbol{a}$ and $\boldsymbol{b}$. Right-hand rule sets direction. Torque in Chapter 11 uses this idea: $\boldsymbol{\tau} = \boldsymbol{r}\times\boldsymbol{F}$.

Before Figure B.3, visualize area and normal as two sides "sweeping" a sheet and a thumb pointing up.

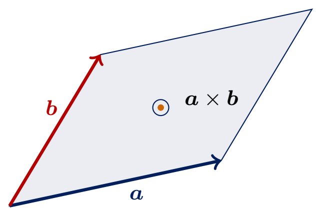


Figure B.3: Area and normal: $\|\boldsymbol{a}\times\boldsymbol{b}\|$ equals the parallelogram area; direction by the right-hand rule.

B.4 Rotations and Change of Basis (2D)

Definition 4 (Matrix)**.** A (real) matrix of size $m\times n$ is a rectangular array of real numbers

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.$$

It represents the linear map that sends a column vector $\boldsymbol{x}\in\mathbb{R}^n$ to $A\boldsymbol{x}\in\mathbb{R}^m$.

In the plane, rotation by angle θ is

$$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \qquad \boldsymbol{a}' = R(\theta)\,\boldsymbol{a}.$$

Geometrically, $R(\theta)$ turns every arrow by θ (angles in *radians* for calculus). These orthonormal rotations preserve lengths and dot products; $\det R(\theta) = 1$. Change of basis simply means describing the same arrow with a different set of unit vectors—numbers change, the arrow does not.

Before Figure B.4, note how both the components and the drawn arrow pivot together; keep labels away from overlap.

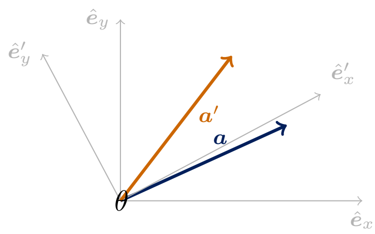


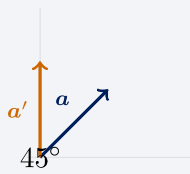
Figure B.4: 2D rotation: $\boldsymbol{a}' = R(\theta)\,\boldsymbol{a}$. The arrow and the basis rotate together; components change accordingly.

Worked Example: 2D Rotation via Matrix

Rotate $\boldsymbol{a} = (1, 1)$ by $\theta = 45°$. With

$$R(\tfrac{\pi}{4}) = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}, \quad R(\tfrac{\pi}{4})\,\boldsymbol{a} = \begin{bmatrix} \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} \end{bmatrix} = \begin{bmatrix} 0 \\ \sqrt{2} \end{bmatrix}.$$

The vector turns by $45°$ to lie on the positive y-axis with length $\|\boldsymbol{a}\| = \sqrt{2}$ preserved.

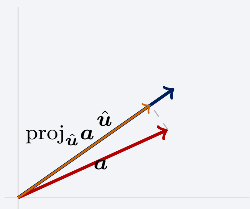


Worked Example: Projection Length

Let $\boldsymbol{a} = (3, 1)$ and $\hat{\boldsymbol{u}} = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$. The projection length of $\boldsymbol{a}$ onto $\hat{\boldsymbol{u}}$ is

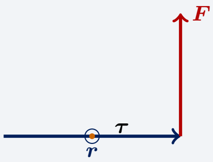
$$\boldsymbol{a} \cdot \hat{\boldsymbol{u}} = 3 \cdot \frac{1}{\sqrt{2}} + 1 \cdot \frac{1}{\sqrt{2}} = \frac{4}{\sqrt{2}} = 2\sqrt{2},$$

so the projected vector is $(2\sqrt{2})\,\hat{\boldsymbol{u}}$.



Worked Example: Area and Torque

Let $\boldsymbol{a} = (2, 0, 0)$ and $\boldsymbol{b} = (0, 1, 0)$. The parallelogram area is $\|\boldsymbol{a} \times \boldsymbol{b}\| = \|(0, 0, 2)\| = 2$. If $\boldsymbol{r} = (2, 0, 0)$ m and $\boldsymbol{F} = (0, 5, 0)$ N, then $\boldsymbol{\tau} = \boldsymbol{r} \times \boldsymbol{F} = (0, 0, 10)$ N · m points along the $+z$ axis (right-hand rule).



Common Pitfalls

Mixing up magnitudes and components; forgetting unit vectors in projections; using degrees in formulas that expect radians; right-hand rule sign errors for a cross product; thinking a change of basis changes the underlying arrow.

Try in 60 seconds

Quick checks:

- **Quick dot.** Compute $\boldsymbol{a} \cdot \boldsymbol{b}$ for $\boldsymbol{a} = (3, 1)$, $\boldsymbol{b} = (2, 2)$ and interpret the sign.
- **Shadow length.** For $\boldsymbol{a} = (1, 2)$ on $\hat{\boldsymbol{u}} = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$, find $\mathrm{proj}_{\hat{\boldsymbol{u}}}\boldsymbol{a}$.
- **Right-hand check.** If $\boldsymbol{a} = (1, 0, 0)$ and $\boldsymbol{b} = (0, 1, 0)$, what is $\boldsymbol{a} \times \boldsymbol{b}$ and along which axis does it point?

Appendix C

Multivariable Calculus Lite

We collect just the multivariable ideas needed for fields and potentials: gradient (steepest ascent), directional derivatives (rate along a direction), and a light touch on divergence, curl, and line integrals.

Learning Objectives

You can read gradients as arrows pointing uphill, compute directional derivatives, recognize conservative fields and recover a potential, and interpret divergence/curl qualitatively.

Symbols at a Glance

∇ gradient operator; ∇f gradient of scalar f; $\nabla \cdot \boldsymbol{F}$ divergence; $\nabla \times \boldsymbol{F}$ curl; $\mathrm{d}\boldsymbol{\ell}$ line element; ϕ potential.

Analogy: Height Map

Think of $f(x, y)$ as a landscape. The gradient is the arrow pointing straight uphill; its length is how steep the slope is at your feet.

C.1 Gradient and Directional Derivative

For a scalar field $f : \mathbb{R}^n \to \mathbb{R}$, the gradient

$$\nabla f = \begin{bmatrix} \partial f/\partial x_1 \\ \vdots \\ \partial f/\partial x_n \end{bmatrix}$$

points in the direction of steepest increase. The directional derivative along a unit vector $\hat{\boldsymbol{u}}$ is

$$D_{\hat{\boldsymbol{u}}} f = \nabla f \cdot \hat{\boldsymbol{u}},$$

the instantaneous rate of change of f in the $\hat{\boldsymbol{u}}$ direction.

Before Figure C.1, remember: longer arrows mean steeper uphill.

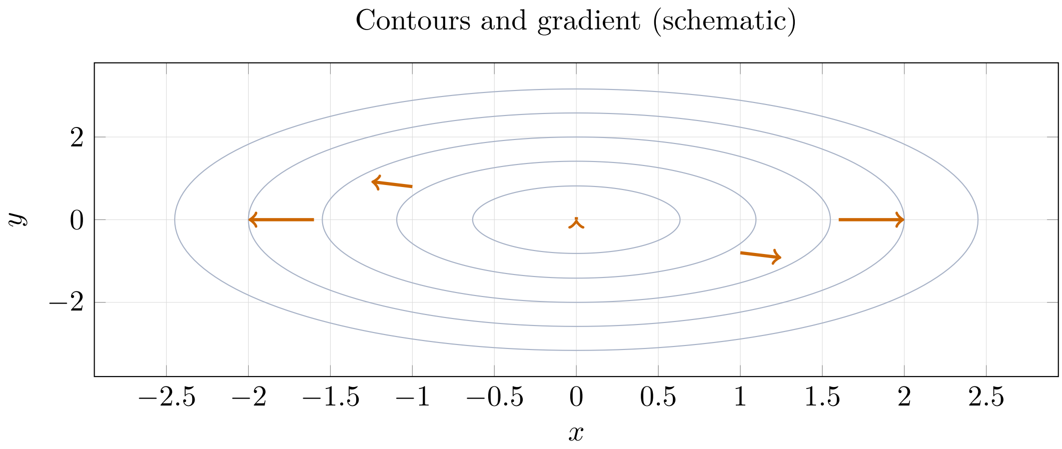


Figure C.1: Contours of a bowl-shaped function with a few gradient arrows: they point uphill and grow with steepness.

Rays in a Graded Index (Qualitative)

In media where the refractive index $n(x, y)$ varies smoothly, light rays bend toward higher n. A compact vector statement is $\frac{\mathrm{d}}{\mathrm{d}s}(n\,\hat{\boldsymbol{s}}) = \nabla n$, where $\hat{\boldsymbol{s}}$ is the unit tangent along the ray and s is arclength. You do not need to compute this here—read it as "rays drift toward increasing n."

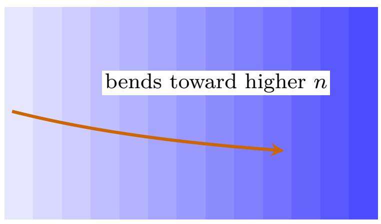


Figure C.2: A graded index: shading suggests higher n to the right; a ray curves toward that side.

C.2 Conservative Fields and Potentials

A vector field $\boldsymbol{F}$ is conservative if it is the gradient of a potential ϕ: $\boldsymbol{F} = \nabla\phi$ (potential defined up to an additive constant). Then line integrals are path-independent and

$$\int_{\mathcal{C}} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} = \phi(\text{end}) - \phi(\text{start}).$$

In 2D/3D on simple regions, a sufficient test is that $\nabla \times \boldsymbol{F} = \boldsymbol{0}$ and the region is simply connected. In Chapter 8, this is the condition for a force to admit a potential energy; in Chapter 12, $\boldsymbol{F} = -\nabla U$ with $U = -GMm/r$.

C.3 Divergence and Curl (Qualitative)

The divergence $\nabla \cdot \boldsymbol{F}$ measures sources and sinks; positive divergence looks like fluid expanding from a point. The curl $\nabla \times \boldsymbol{F}$ measures local rotation: a nonzero curl makes tiny paddles spin.

Before Figure C.3, keep the pictures in mind: arrows spreading vs. arrows swirling.

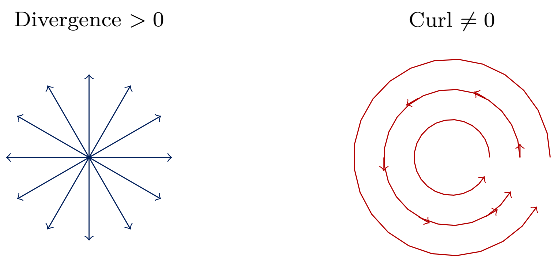


Figure C.3: Left: arrows spreading out (positive divergence). Right: arrows circling (nonzero curl).

C.4 Line Integrals Along Paths

For a path $\mathcal{C}\colon \boldsymbol{r}(t)$, $a \le t \le b$, the line integral of a vector field is

$$\int_{\mathcal{C}} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} = \int_a^b \boldsymbol{F}(\boldsymbol{r}(t)) \cdot \boldsymbol{r}'(t)\, \mathrm{d}t.$$

If $\boldsymbol{F} = \nabla\phi$, the integral depends only on endpoints. Orientation matters: reversing the path flips the sign.

Common Pitfalls

Confusing gradient direction with the direction to a particular point; forgetting to normalize the direction in a directional derivative; assuming $\nabla \times \boldsymbol{F} = \boldsymbol{0}$ everywhere automatically makes a field conservative on domains with holes.

Try in 60 seconds

Quick checks:

- **Quick grad.** For $f(x, y) = x^2 + 3y^2$, compute ∇f at $(1, 1)$ and the directional derivative along $(1, 1)/\sqrt{2}$.
- **Endpoint check.** If $\boldsymbol{F} = \nabla\phi$ and a path goes from $(0, 0)$ to $(2, 1)$, which values matter to the line integral?
- **Swirl or source?** Which picture indicates curl, which indicates divergence?

Appendix D

Ordinary Differential Equations Essentials

We keep ODE tools lean and visual: how to read direction, how to separate variables, how integrating factors tame linear equations, and how second-order linear models (oscillators) behave.

Learning Objectives

You can solve simple separable and linear first-order ODEs, interpret phase portraits for second-order systems, and connect forcing/damping with resonance and decay.

Symbols at a Glance

$y' = \dfrac{\mathrm{d}y}{\mathrm{d}t}$; $\dot{x}, \ddot{x}$ time derivatives; parameters k, γ, ω_0 for growth/damping/natural frequency; ω driving frequency.

Analogy: Slope Field as Grass Arrows

Imagine a hillside covered with tiny arrows showing the local slope. A solution is a path that always follows the arrows.

D.1 Reading Solutions from Direction (Slope) Fields

For a first-order ODE $y' = f(t, y)$, a slope field sketches a short line segment of slope $f(t, y)$ at each grid point. Solutions trace curves tangent to these segments.

Before Figure D.1, note: for the logistic law $y' = y(1 - y)$, slopes depend only on y; below $y = 0$ arrows point downward (negative), between 0 and 1 they tilt upward, and near $y = 1$ they flatten.

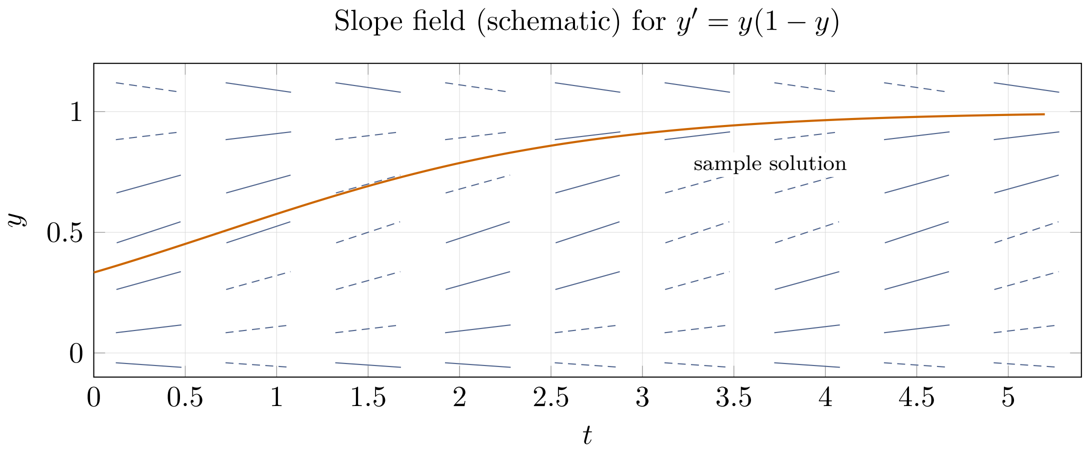


Figure D.1: Direction field for the logistic law: arrows tilt upward for $0 < y < 1$ and flatten near the carrying capacity $y = 1$.

D.2 Separable and Linear First-Order ODEs

Separable: if $y' = g(t)h(y)$, rearrange $\dfrac{\mathrm{d}y}{h(y)} = g(t)\,\mathrm{d}t$ and integrate both sides. Linear: if $y' + p(t)y = q(t)$, multiply by an integrating factor $\mu(t) = e^{\int p(t)\,dt}$ to make $(\mu y)' = \mu q$.

Worked Example: Growth with Carrying Capacity

Solve $y' = r\,y(1-y)$ for $y(0) = y_0$ (units scaled so the capacity is 1). Separating variables gives

$$\int \frac{\mathrm{d}y}{y(1-y)} = \int r\,\mathrm{d}t \quad \Rightarrow \quad \ln\frac{y}{1-y} = rt + C.$$

Thus $\dfrac{y}{1-y} = Ae^{rt}$ with $A = \dfrac{y_0}{1-y_0}$ and

$$\boxed{y(t) = \frac{1}{1 + \dfrac{1-y_0}{y_0}e^{-rt}}}$$

which approaches 1 as $t \to \infty$, matching the slope field in Figure D.1.

Worked Example: Integrating Factor

Solve $y' + 2y = e^{-t}$, $y(0) = 0$. The integrating factor is $\mu(t) = e^{\int 2\,dt} = e^{2t}$. Then

$$\frac{d}{dt}\left(e^{2t}y\right) = e^{2t}e^{-t} = e^{t}$$

$$\Rightarrow$$

$e^{2t}y = e^t + C$, and with $y(0) = 0$ we find $C = -1$. Therefore $\boxed{y(t) = e^{-t} - e^{-2t}}$.

D.3 Second-Order Linear ODEs: Oscillations

The undamped oscillator $\ddot{x} + \omega_0^2 x = 0$ has sinusoidal solutions. Damping adds decay: $\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = 0$. Forced response adds a drive $F_0 \cos\omega t$ and exhibits resonance near $\omega \approx \omega_0$ when damping is small.

Before Figure D.2, recall: phase portraits plot $(x, \dot{x})$; undamped orbits are closed; damping spirals inward.

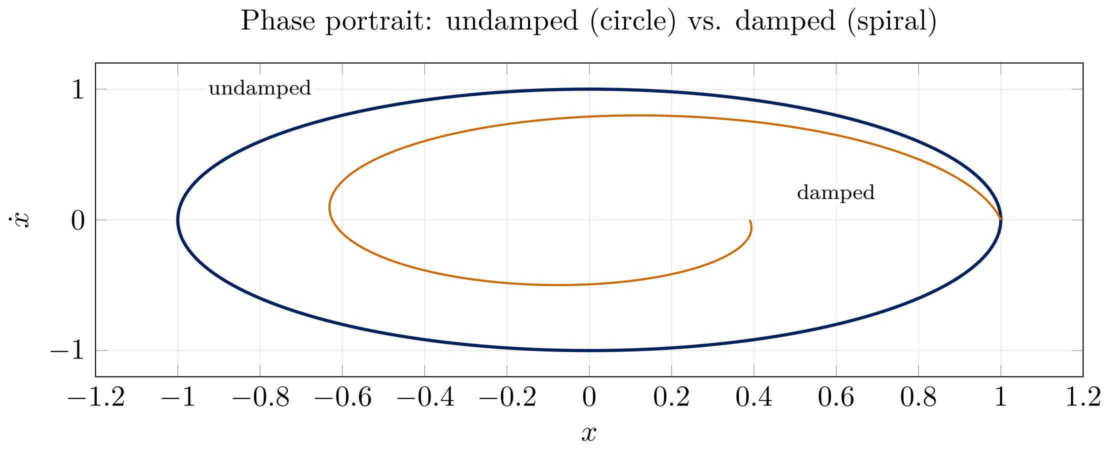


Figure D.2: Closed orbits (undamped) become inward spirals with damping; arrows would point counterclockwise for $\ddot{x} + \omega_0^2 x = 0$.

Before Figure D.3, keep in mind: damping flattens and broadens the resonance peak.

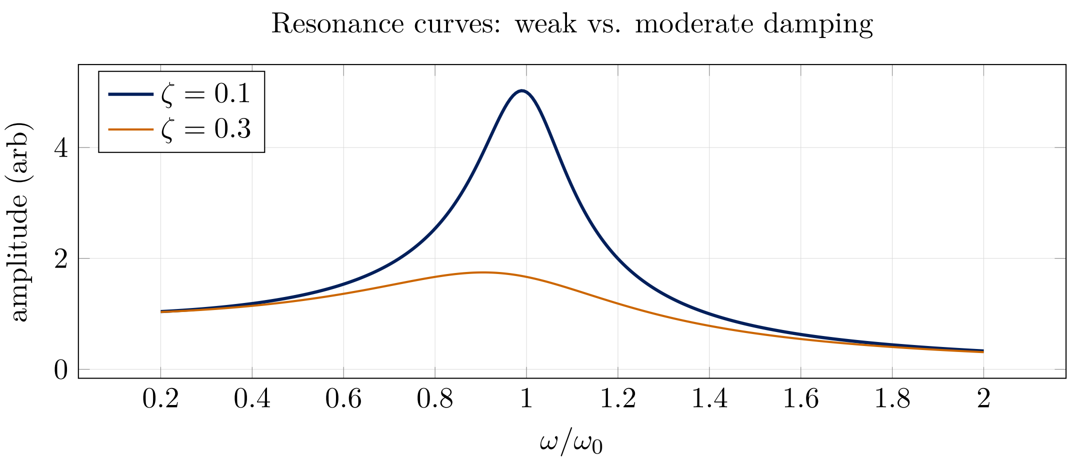


Figure D.3: Forced oscillator amplitude (dimensionless) vs. drive frequency ratio $\Omega = \omega/\omega_0$ for two damping ratios ζ.

D.4 Existence and Uniqueness (Statement)

If f and $\partial f/\partial y$ are continuous near (t_0, y_0) (Lipschitz in y suffices), then the initial value problem $y' = f(t, y)$, $y(t_0) = y_0$ has a unique solution in some interval around t_0. Practically: well-behaved right-hand sides give a single trajectory through each point; discontinuities or non-Lipschitz points can lead to multiple or no solutions.

Common Pitfalls

Forgetting to divide by a function that can be zero when separating variables; dropping the constant of integration; confusing transient (decaying) with steady-state (driven) parts; reading phase portraits without units or axes.

Try in 60 seconds

Quick checks:

- **Separate it.** Solve $y' = 3y$ with $y(0) = 2$.
- **IF quick.** Solve $y' + y = 1$; what is the long-time limit?
- **Read the portrait.** On Figure D.2, is the motion clockwise or counterclockwise? What changes if the sign in $\ddot{x} + \omega_0^2 x = 0$ flips?

Appendix E

Numerical Methods Quick Reference

This appendix distills the discrete tools used across the book into one friendly place. Our aim is intuition you can act on: what each update does, how accuracy improves with smaller steps, and how to sanity-check stability and energy behavior.

Learning Objectives

You can write and compare explicit vs. symplectic Euler updates, estimate error vs. step size on a log–log plot, and spot energy drift vs. boundedness in conservative systems.

Symbols at a Glance

Δt step; $t_n = n\Delta t$; $x_n \approx x(t_n)$; $v_n \approx v(t_n)$; local error $\mathcal{O}(\Delta t^{p+1})$, global error $\mathcal{O}(\Delta t^p)$ for order p.

Analogy: Shutter Speed

An integrator is like a camera filming motion. A large shutter time (big Δt) makes blurry frames and can miss fast wiggles; a small Δt makes sharp frames but needs more battery. Your goal is crisp enough without draining the battery.

E.1 Finite Differences and Euler Updates

For $\dot{x} = f(x,t)$ at t_n, the forward difference reads

$$\frac{x_{n+1} - x_n}{\Delta t} \approx f(x_n, t_n),$$

meaning "new minus old equals slope times step." In mechanics we evolve the pair (x, v) via $\dot{x} = v$ and $\dot{v} = a(x, v, t)$:

$$\begin{aligned}
\text{Explicit Euler:} \quad & v_{n+1} = v_n + a(x_n, v_n, t_n)\,\Delta t, \\
& x_{n+1} = x_n + v_n\,\Delta t. \\
\text{Symplectic Euler:} \quad & v_{n+1} = v_n + a(x_n, v_n, t_n)\,\Delta t, \\
& x_{n+1} = x_n + v_{n+1}\,\Delta t.
\end{aligned}$$

"Order" counts how fast error shrinks as you shrink Δt. Explicit Euler is first-order and simple. Symplectic Euler is also first-order but uses the fresh velocity to update position, which makes a big qualitative difference for conservative systems: it tends to bound energy instead of drifting.

Worked Example: One SHO Step

For $\ddot{x} = -x$ with $(x_0, v_0) = (1, 0)$ and $\Delta t = 0.1$:

- Explicit Euler: $a_0 = -x_0 = -1$, so $v_1 = v_0 + a_0\Delta t = -0.1$, $x_1 = x_0 + v_0\Delta t = 1.0$.
- Symplectic Euler: same $v_1 = -0.1$, but $x_1 = x_0 + v_1\Delta t = 0.99$ (uses the new v).

After several steps, explicit Euler's energy creeps; symplectic's stays near constant.

E.2 Error vs. Step Size

On log–log axes, first-order global error falls with slope 1. A practical recipe is to run with Δt and with $\Delta t/2$; the observed change estimates the error size. Before Figure E.1, keep that slope-1 picture in mind: halving Δt roughly halves the error for order-1 schemes.

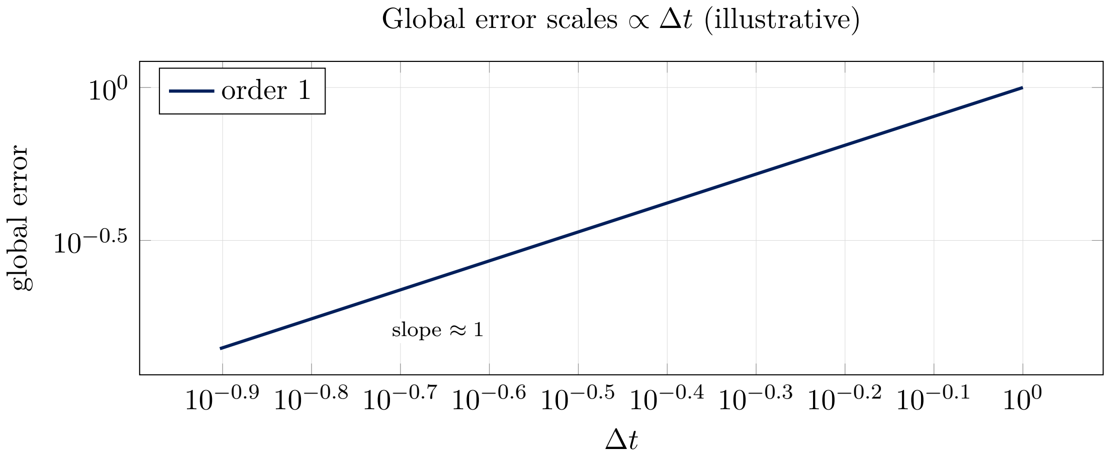


Figure E.1: Illustrative error vs. step size on log–log axes.

E.3 Energy Behavior (Conservative Systems)

For $\ddot{x} + \omega_0^2 x = 0$, explicit Euler typically drifts in energy (frames "gain" or "lose" energy), while symplectic Euler tends to oscillate around the constant true energy. When in doubt for long conservative runs, pick the symplectic flavor.

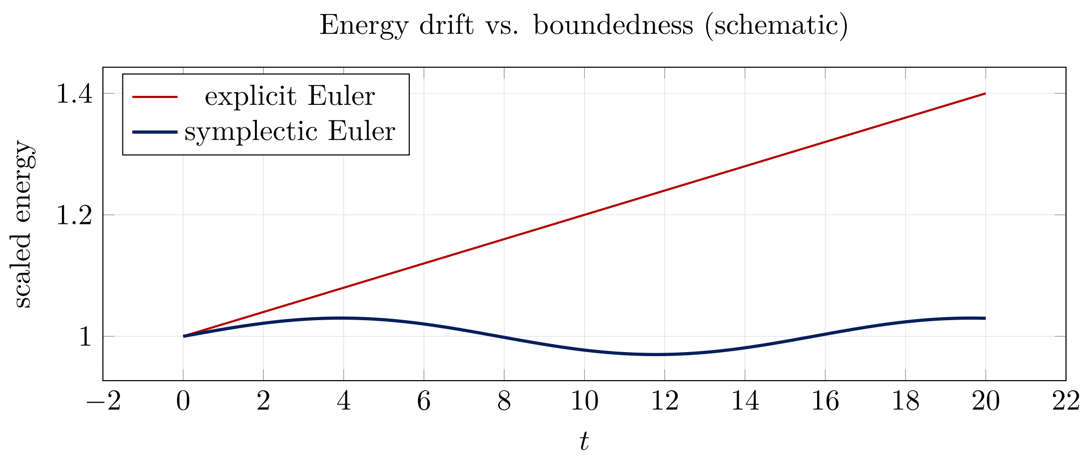


Figure E.2: Explicit Euler often drifts; symplectic Euler keeps energy bounded around the truth.

E.4 Stability Sketches

For the test equation $y' = \lambda y$ (the standard stability yardstick), explicit Euler is stable only if $|1 + \lambda\,\Delta t| < 1$. For real $\lambda < 0$, this collapses to $-2 < \lambda\,\Delta t < 0$—too large a step turns decay into blow-up.

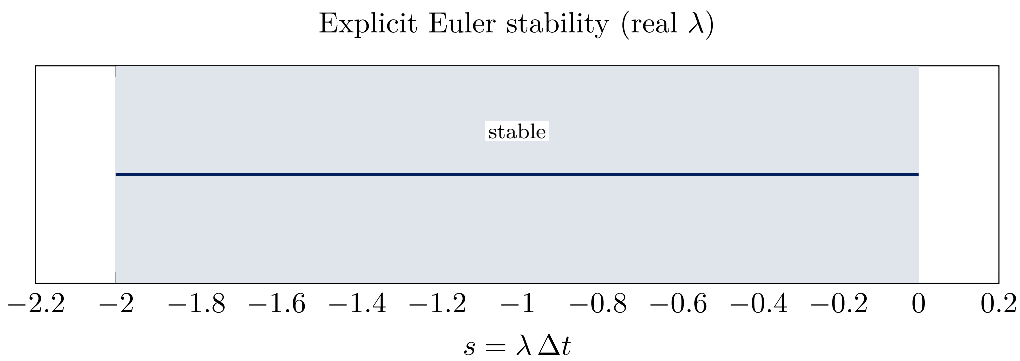


Figure E.3: On the real axis, explicit Euler is stable for $-2 < \lambda\,\Delta t < 0$.

When you lack a trusted reference answer, halve Δt and check that results change by the expected order and fall below a tolerance you set up front (units matter!).

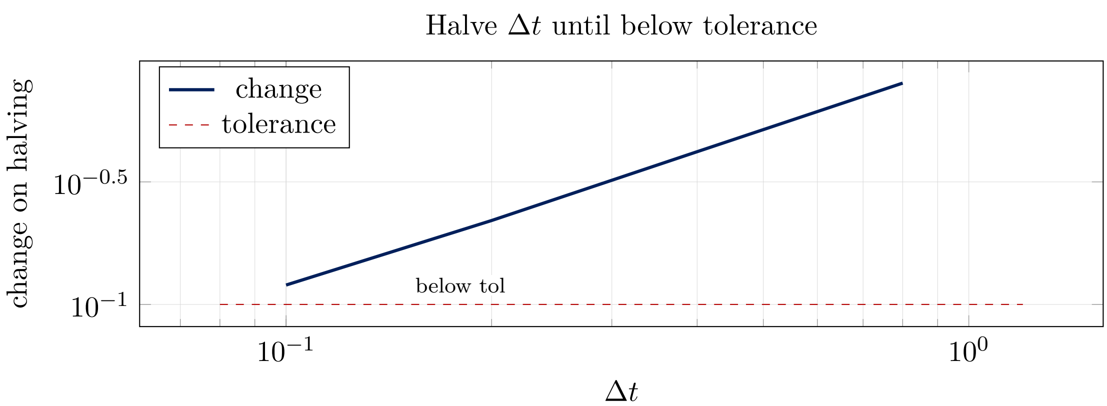


Figure E.4: A practical convergence probe: reduce Δt until changes fall below a set tolerance.

E.5 Quick Checks

A few habits that pay off in practice:

- Reduce Δt and see if answers change by $\mathcal{O}(\Delta t)$ (order-1) or faster.
- In conservative tests, monitor energy; prefer symplectic Euler for long runs.
- Keep units consistent; errors in scale can masquerade as “instability.”

Choosing a Stepper: Quick Pros/Cons

When in doubt, this short list helps you pick and set expectations:

- **Explicit Euler** — simple, cheapest per step; first-order; can drift in invariants (energy); tight stability limits (Figure E.3).
- **Symplectic Euler** — same cost/order; uses new velocity; much better qualitative energy behavior in conservative systems (Figure E.2).
- **Defaults** — mechanics with conserved energy: start symplectic; strongly damped/short transients: either works with small enough Δt.
- **Safety** — always halve Δt once to gauge error (Figure E.4); watch units and scales.

Common Pitfalls

Confusing local and global error; trusting a single Δt; updating x with the old v when intending symplectic Euler; hidden unit mismatches.

Try in 60 seconds

- **Write it.** Write explicit and symplectic Euler steps for $\ddot{x} = -x$.
- **Order check.** If Δt halves and error halves, what order is your method?
- **Energy probe.** Which method would you pick to integrate a planet's nearly circular orbit?

Glossary

Short, alphabetized definitions for quick lookup. Cross-links point to chapters where a concept features prominently.

Acceleration Rate of change of velocity; in 1D $a = \dot{v}$; see Chapter 3.

Angular acceleration Rate of change of angular velocity $\alpha = \dot{\omega}$; see Chapter 11.

Angular momentum Rotational analogue of momentum; conserved in absence of external torque; see Chapter 11.

Area (under a curve) Signed integral; area under $v(t)$ gives displacement; see Chapter 3.

Buoyancy Upward force on a body in a fluid equal to the weight of displaced fluid; see Chapter 14.

Center of mass (COM) Weighted average of position; system moves as if mass were concentrated at COM; see Chapter 10.

Coefficient of restitution Dimensionless measure of bounciness in collisions; ratio of relative speeds after/before; see Chapter 10.

Conservative force One with path-independent work and a potential U; see Chapter 9.

Continuity equation Statement of conservation (e.g., mass) for a flowing medium; see Chapter 14.

Damping ratio Dimensionless measure of damping strength ζ; relates to decay rate and quality factor; see Chapter 13.

Dot product Measure of alignment between vectors; work = $\boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r}$; see Chapter 8.

Divergence Scalar measure of sources/sinks of a vector field; see Appendix C.

Energy Capacity to do work. Kinetic $K = \frac{1}{2}mv^2$; potential U depends on configuration; see Chapters 8 and 9.

Force Interaction that changes motion; Newton's second law $\sum \boldsymbol{F} = m\,\boldsymbol{a}$; see Chapter 7.

Froude number Dimensionless ratio $\mathrm{Fr} = v/\sqrt{gL}$ comparing inertia to gravity in free-surface flows; see Chapter 15.

Gradient Vector of partial derivatives pointing uphill; see Appendix C.

Impulse Integral of force over time; changes momentum; see Chapter 10.

Inertia Resistance to changes in motion; quantified by mass; rotational counterpart is moment of inertia.

Line integral Integral of a vector field along a path; work; see Appendix C and chapter 8.

Matrix Rectangular array representing a linear map; rotations use $R(\theta)$; see Appendix B.

Moment of inertia Rotational inertia about an axis; appears in $\tau = I\alpha$ and $K_{\text{rot}} = \frac{1}{2}I\omega^2$; see Chapter 11.

Momentum Product of mass and velocity $\boldsymbol{p} = m\,\boldsymbol{v}$; conserved for isolated systems; see Chapter 10.

Normal force Contact force perpendicular to a surface; appears in FBDs; see Chapter 7.

Numerical method Discrete scheme to approximate ODE solutions (e.g., Euler, symplectic Euler); see Chapter 16 and appendix E.

Potential energy Stored capability of conservative forces; changes by negative work; see Chapter 9.

Power Rate of doing work $P = \boldsymbol{F} \cdot \boldsymbol{v}$; see Chapter 8.

Projection Component of one vector along another; dot product; see Appendix B.

Quality factor Dimensionless $Q = 1/(2\zeta)$; higher Q means slower decay; see Chapter 13.

Resonance Amplification when driving frequency matches a system's natural frequency; see Chapter 13.

Resultant Net vector sum, e.g., net force; see Chapter 7.

Reynolds number Dimensionless ratio $\text{Re} = \rho v L/\eta$ comparing inertia to viscosity; see Chapter 15.

Rotational kinematics Angular position θ, velocity ω, acceleration α; see Chapter 11.

Slope Geometric meaning of derivative; tangent line slope; see Chapter 2 and appendix A.

Specific mechanical energy Energy per unit mass $\varepsilon = \frac{1}{2}v^2 + \Phi$; negative for bound orbits; see Chapter 12.

Symplectic method Structure-preserving time-stepper for Hamiltonian systems; better long-term energy behavior; see Appendix E.

Torque Tendency to rotate: $\boldsymbol{\tau} = \boldsymbol{r} \times \boldsymbol{F}$; see Chapter 11.

Unit vectors Cartesian basis arrows $\hat{\boldsymbol{e}}_x, \hat{\boldsymbol{e}}_y, \hat{\boldsymbol{e}}_z$ of length 1 pointing along the coordinate axes; see Appendix B.

Vector Quantity with magnitude and direction; added tip-to-tail; see Chapter 5 and appendix B.

Velocity Rate of change of position; see Chapter 3.

Work Energy transfer by force along a path: $W = \int \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r}$; see Chapter 8.

Index of Symbols

Alphabetical list of the most commonly used symbols. Units are indicated in brackets where fixed by context.

A Area; cross-sectional area (fluids, drag) [m^2].

C_D Drag coefficient (dimensionless).

E Total mechanical energy, $E = K + U$.

e Coefficient of restitution (dimensionless).

ε Specific mechanical energy per unit mass [J/kg].

F**,** $\boldsymbol{F}$ Force (scalar magnitude or vector) [N].

Fr Froude number $\mathrm{Fr} = \dfrac{v}{\sqrt{gL}}$ (dimensionless).

g Gravitational acceleration near Earth ($\approx 9.81\,\mathrm{m/s^2}$).

I Moment of inertia about a specified axis [$\mathrm{kg \cdot m^2}$].

K Kinetic energy, $K = \frac{1}{2}mv^2$ [J].

L Angular momentum [$\mathrm{kg \cdot m^2/s}$].

m Mass [kg].

μ, μ_s, μ_k Coefficient(s) of friction (dimensionless; static/kinetic).

μ **(grav.)** Gravitational parameter $\mu = GM$ [$\mathrm{m^3/s^2}$].

p**,** $\boldsymbol{p}$ Linear momentum $\boldsymbol{p} = m\,\boldsymbol{v}$ [$\mathrm{kg \cdot m/s}$].

P Power, $P = \boldsymbol{F} \cdot \boldsymbol{v}$ [W].

Q Quality factor $Q = 1/(2\zeta)$ (dimensionless).

$R(\theta)$ 2D rotation matrix (see Appendix B).

Re Reynolds number $\mathrm{Re} = \dfrac{\rho v L}{\eta}$ (dimensionless).

$\boldsymbol{r} = (x, y, z)$ Position vector [m].

t Time [s]; $t_n = n\,\Delta t$ in numerics.

Δt Time step (see Appendix E).

U Potential energy (context-dependent) [J].

v, $\boldsymbol{v}$ Speed/velocity (scalar/vector) [m/s].

W Work; weight ($W = mg$) by context [J] or [N].

x, y, z Cartesian coordinates [m].

α Angular acceleration [rad/s^2].

η Dynamic viscosity [Pa $\cdot$ s].

ϕ Potential (scalar field) (see Appendix C).

ρ Mass density [kg/m^3].

ζ Damping ratio (dimensionless).

θ Angle (radians unless stated).

τ, $\boldsymbol{\tau}$ Torque [N $\cdot$ m].

ω Angular velocity [rad/s].

∇ Gradient operator; ∇f, $\nabla \cdot \boldsymbol{F}$, $\nabla \times \boldsymbol{F}$ (see Appendix C).

$\hat{\boldsymbol{e}}_x, \hat{\boldsymbol{e}}_y, \hat{\boldsymbol{e}}_z$ Cartesian unit vectors.

Bibliography and Notes

This compact bibliography favors readable, widely available sources. Each item includes a short note on scope and style; historical remarks highlight how ideas entered the canon.

Introductory and Bridge Texts

- **D. Morin, *Introduction to Classical Mechanics.*** Problem-driven, witty, with careful solutions. Excellent for building intuition through practice.
- **A. P. French, *Newtonian Mechanics.*** Clear prose and physical insight; a classic bridge between conceptual understanding and calculation.
- **R. Feynman, *The Feynman Lectures on Physics*, Vol. I.** Big-picture viewpoints and crisp derivations; especially good for energy and conservation ideas.

Standard Undergraduate Texts

- **D. Kleppner and R. Kolenkow, *An Introduction to Mechanics.*** Thorough, with challenging problems; a common reference for momentum and rotation.
- **J. R. Taylor, *Classical Mechanics.*** Gentle exposition with modern notation; accessible treatments of oscillations and central forces.

Historical Sources (Short Remarks)

- **Galileo Galilei,** early kinematics (inclined planes) established constant-acceleration motion and the decomposition of trajectories.
- **Isaac Newton, *Philosophiæ Naturalis Principia Mathematica.*** Unified terrestrial and celestial motion; Newton's laws and the inverse-square gravitation law.
- **Leonhard Euler.** Formalized dynamics and introduced variational and analytical methods that underpin modern mechanics and numerics.